Homework 3

Chapter 11.8. :22

22. $a_n = \frac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}|x-4| = |x-4| .$$

By the Ratio Test, the series converges when $|x-4|<1 \Leftrightarrow -1<x-4<1 \Leftrightarrow 3<x<5$.

When $|x-4|=1$, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^3+1}$, which converges by comparison with the convergent p – series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $(p=2>1)$. Thus, $I=[3,5]$.

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39. By Example 7, $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ for $|x|<1$. In particular, for $x=\frac{1}{\sqrt{3}}$, we have

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{3}\right)^n\frac{1}{\sqrt{3}}\frac{1}{2n+1}$$, so

$$\pi = \frac{6}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} .$$

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48.

$$\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0}\frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots} = \frac{\frac{1}{2}-0}{-\frac{1}{2}-0} = -1$$

since power series are continuous functions.