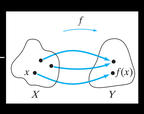
單變數純量函數極限與連續

函數

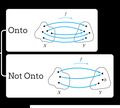
這邊重新幫大家複習函數的觀念，有助於之後學習。

- **Definition**
  Let X and Y be given sets. A function $f: X \to Y$ consists of two sets X and Y together with a "rule" that assigns to each $x \in X$ a special element of Y denoted by $f(x)$. One writes $f(x)$ to denote that x is mapped to the element $f(x)$.
  X is called the domain（定義域）of f, and Y is called the target or co-domain of f.
  The range（值域）of f or the image of f, is the subset of Y defined by $f(X) = \{f(x) \mid x \in X\}$.

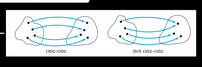
  

  P.S. We will frequently use the notation $f: X \to Y$ for a function.

- **Onto (Surjective)** A funcion $f: X \to Y$ is said to be **onto** if range $f = Y$.
  - Onto
  - Not Onto

- **One-One (Injective)** A function $f: X \to Y$ is called **one-one** if whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

- **Bijection** A funcion $f: X \to Y$ is called an bijection if it is one-to-one and onto.

- **Composite Function**
  Let X, Y, and Z be sets and $f: X \to Y$ and $g: Y \to Z$ be functions.
  By following f with g, we obtain a function $g \circ f: X \to Z$ called the composite of g and f. Thus $(g \circ f)(x) = g(f(x)) \ \forall x \in X$
  Functional composition is associative but not commutative.

- **Invertible Function**
  A function $f: X \to Y$ is said to be invertible if $\exists\, g: Y \to X$ such that $\begin{cases} (f \circ g)(y) = y & \forall y \in Y \\ (g \circ f)(x) = x & \forall x \in X \end{cases}$
  If such a function g exists, then it is unique and is called the inverse of f. We denote the inverse of f (when it exists) by f^{-1}.

  Theorem
  1. f is invertible $\Leftrightarrow$ f is bijection
  2. If $f: X \to Y$ is invertible, then f^{-1} is invertible, and $(f^{-1})^{-1} = f$.
  3. If $f: X \to Y$ and $g: B \to C$ are invertible, then $g \circ f$ is invertible, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

極限

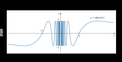
- **定義**
  Let f be a function defined on some open interval that contains the number a, except possibly at a itself. We write $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.
  若極限值存在則唯一

- **單邊極限**
  - Left-hand Limit
    $\lim_{x \to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$
  - Right-hand Limit
    $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$

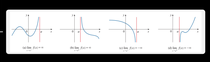
  $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

- **不存在的情況**
  - 左右極限不同

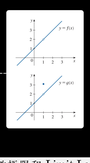
  - 震盪

  - Infinite Limit
    - 定義
      $\lim_{x \to a} f(x) = \infty : \forall M > 0, \exists \delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > M$
      $\lim_{x \to a} f(x) = -\infty : \forall N < 0, \exists \delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < N$
    - 垂直漸近線
      The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true :
      $\lim_{x \to a} f(x) = \infty$ $\quad \lim_{x \to a_-} f(x) = \infty$ $\quad \lim_{x \to a_+} f(x) = \infty$
      $\lim_{x \to a} f(x) = -\infty$ $\quad \lim_{x \to a_-} f(x) = -\infty$ $\quad \lim_{x \to a_+} f(x) = -\infty$

- **Proposition**
  - 極限的局部有限性
    Suppose that $\lim_{x \to a} f(x) = L$, then $\exists\, \delta > 0,\ M > 0$ such that if $0 < |x - a| < \delta$ then $|f(x)| < M$.
  - If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limit exists.

  - Limit Laws
    Suppose that c is a constant and the limits $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.
    - $\lim_{x \to a} [f(x) \pm g(x)] = L \pm M$
    - $\lim_{x \to a} [\, cf(x)\,] = cL$
    - $\lim_{x \to a} [f(x)g(x)] = LM$
    - $\lim_{x \to a} [f(x)]^n = L^n$ where $n \in \mathbb{N}$
    - $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \begin{cases} \frac{L}{M} & \text{if } M \neq 0 \text{ and } L \neq 0 \\ \text{doesn't exist} & \text{if } M = 0 \end{cases}$
    - $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ where $n \in \mathbb{N}$
      If n is even, we assume that $L > 0$.

    透過已知函數極限和 Limit Laws 可以更快地算出未知函數的極限！例如，透過 Limit Laws，我們可以知道 If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

- **Theorem**
  - Squeeze Theorem
    Let $p > 0$. Suppose that, for all x such that $f(x) \leq g(x) \leq h(x)$ when $0 < |x - c| < p$.
    If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

- **Limit at Infinity**
  - 定義
    - $\lim_{x \to \infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that if $x > N$ then $|f(x) - L| < \varepsilon$
    - $\lim_{x \to -\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that if $x < N$ then $|f(x) - L| < \varepsilon$

    若極限存在則唯一！
    All the Limit Laws are true when we replace $\lim_{x \to c}$ by $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$.
    That is, the variable x may approach a finite number c or $\pm\infty$.
  - Theorem
    1. If $r > 0$ is a rational number and $c \in \mathbb{R}$, then $\lim_{x \to \infty} \dfrac{c}{x^r} = 0$.
    2. If $r > 0$ is a rational number such that x^r is defined for all x and $c \in \mathbb{R}$, then $\lim_{x \to -\infty} \dfrac{c}{x^r} = 0$.

    這個定理在計算無窮遠處的極限時很好用，證明也不難，同學們可以自行嘗試證明。
  - 漸近線
    - Definition
      The straight line $y = mx + k$ is an asymptote of the graph $y = f(x)$ if $\lim_{x \to \infty} (f(x) - mx - k) = 0$ or $\lim_{x \to -\infty} (f(x) - mx - k) = 0$
      The straight line $y = mx + k$ is called a horizontal asymptote of the graph of f if $m = 0$, and is called a slant asymptote of the graph of f if $m \neq 0$.
    - 計算方法
      If $y = mx + k$ is a slant asymptote of the graph $y = f(x)$, then $m = \lim_{x \to \infty} \dfrac{f(x)}{x}$ or $m = \lim_{x \to -\infty} \dfrac{f(x)}{x}$
      and $k = \lim_{x \to \infty} (f(x) - mx)$ or $k = \lim_{x \to -\infty} (f(x) - mx)$.

      水平漸近線的定義可以寫成—The line $y = L$ is a horizontal asymptote of curve $y = f(x)$ if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

      小技巧 若曲線 $y = f(x)$ 可表為 $y = mx + k + g(x)$ 且 $\lim_{x \to \pm\infty} g(x) = 0$，則此曲線具有 $y = mx + k$ 的漸近線！
    - 觀念補充
      若 $\lim_{x \to \infty} f(x) = L$ 存在，則 $m = \lim_{x \to \infty} \dfrac{f(x)}{x} = 0$。
      若 $m = \lim_{x \to \infty} \dfrac{f(x)}{x}$ 存在且不等於 0，則 $\lim_{x \to \infty} f(x) = \pm\infty$。
      綜合上述，我們知道水平和斜漸近線不會並存，也就是只存在其中一個或都不存在。（上述的 ∞ 可改成 $-\infty$）
  - Infinite Limit at Infinity
    - $\lim_{x \to \infty} f(x) = \infty : \forall M > 0, \exists N > 0$ such that if $x > N$ then $f(x) > M$
    - $\lim_{x \to \infty} f(x) = -\infty : \forall M < 0, \exists N > 0$ such that if $x > N$ then $f(x) < M$
    - $\lim_{x \to -\infty} f(x) = \infty : \forall M > 0, \exists N < 0$ such that if $x < N$ then $f(x) > M$
    - $\lim_{x \to -\infty} f(x) = -\infty : \forall M < 0, \exists N < 0$ such that if $x < N$ then $f(x) < M$

- **重要極限**
  - $\lim_{x \to a} 1 = 1$
  - $\lim_{x \to a} x = a$
  - $\lim_{x \to a} |x| = |a|$
  - $\lim_{x \to 0} \dfrac{\sin ax}{ax} = 1$ for each number $a \neq 0$
  - $\lim_{x \to \infty} \dfrac{\sin ax}{ax} = 0$ for each number $a \neq 0$
  - $\lim_{x \to 0} \dfrac{1 - \cos ax}{ax} = 0$ for each number $a \neq 0$
  - $\lim_{x \to \infty} \dfrac{\cos ax - 1}{ax} = 0$ for each number $a \neq 0$

連續

- **定義**
  - 在點上連續 A function f is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$
  - 單邊連續
    - A function f is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$
    - A function f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$

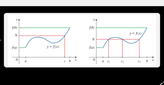
    f is continuous at a iff $f(a)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$ all exist and are equal.
  - 在區間上連續 A function f is continuous on an interval if it is continuous at every number in the interval.
    (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)
  - 處處連續 A function f is continuous everywhere if it is continuous at its domain.

- **不連續的情況**

- **Proposition**
  If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :
  1. $f \pm g$ 2. cf 3. fg 4. $\dfrac{f}{g}$ if $g(a) \neq 0$
  - If f is continuous at b, and $\lim_{x \to a} g(x) = b$ then $\lim_{x \to a} f(g(x)) = f(b)$. In other words, $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

    Corollary If f is continuous at $g(a)$ and g is continuous at a $\left(\lim_{x \to a} g(x) = g(a)\right)$, then $\lim_{x \to a} f(g(x)) = f(g(a))$. That is $f(g(x))$ is continuous at a.

- **Theorem**
  - Intermediate Value Theorem
    Suppose that g is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$.
    $\Rightarrow$ Then there exists a number c in (a, b) such that $f(c) = N$.

    Special Case

    Balzano's Theorem 勘根定理 If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.
  - Extreme Value Theorem
    If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.
    (連續函數在閉區間上必有最大最小值)

- **在定義域上一定連續的函數**
  - Polynominal Function
  - Rational Function
  - Root Function
  - Logarithmic Function
  - Exponential Function
  - Trigonometric Function
  - Inverse Trigonometric Function

單變數純量函數導數（概念篇）

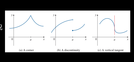
定義

A function is said to be differentiable at a if $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists.
If the limit exists, it is called the derivative of f at a and is denoted by $f'(a)$.

A function is differentiable on (a,b) if it is differentiable at every number in the interval.
We also say that f is differentiable if f is differentiable at a for every a in the domain of f.

Tangent Line If f is differentiable at a, we define the tangent line to the graph of f at $(a, f(a))$ to be the line through $(a, f(a))$ with slope $f'(a)$. As an equation for this line we can write $y - f(a) = f'(a)(x-a)$.

導數不存在的情況（不可微）



The graph of f in (c) is not differentiable at a. For reasons geometrically evident, We say that the graph of f in (c) has vertical tangent at a.

The Derivative as a Function

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Other Notations $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$ The symbols D and $\frac{d}{dx}$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

Higher Derivative

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}}\underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Derivative of Some Functions

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}(a^x) = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x\lim_{h\to 0}\frac{a^h-1}{h} = f'(0)a^x$$

$$\lim_{h\to 0}\frac{a^h-1}{h} = f'(0)$$

自然指數其中一種定義 e is the number such that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$, which means $\frac{d}{dx}e^x = e^x$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\csc x) = -\csc x\cot x$

$\frac{d}{dx}(\cos x) = -\sin x$ $\quad$ $\frac{d}{dx}(\sec x) = \sec x\tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

Proposition

可微必連續

Let f be defined on (a,b). If f is differentiable at a point $x \in (a,b)$, then f is continuous at x.

Differentiation Rules

Suppose that c is a constant and f and g are both differentiable.

$$\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)^2]}$$

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$,
then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x
and F' is given by the product $F'(x) = f'(g(x))\cdot g'(x)$
In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions,
then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

範例1 If $n \in \mathbb{R} - \{0\}$ and $u = g(x)$ is differentiable, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$
Alternatively, $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\cdot g'(x)$.

範例2
$$\frac{d}{dx}[a^{g(x)}] = \frac{d}{dx}[e^{(\ln a)g(x)}] = e^{(\ln a)g(x)}(\ln a)g'(x) = a^{g(x)}(\ln a)g'(x)$$

Implicit Differentiation

Differentiable both side of the equation with respect to x.

多變數的 Chain Rule 可以讓明隱函數定理！不過因為還沒學到多變數，所以先講做法。

範例 For $x^2+y^2=1$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$
$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} \quad \left(\frac{dy}{dx} = -\frac{x}{y}\right) = -\frac{y+\frac{x^2}{y}}{y^2} = -\frac{1}{y} - \frac{x^2}{y^3}$$

補充：隱函數定義 An implicit function is a function that is defined implicitly by an equation that x and y satisfy, by associating one of the value y with the arguments x.
For example, $x^2+y^2=1$ and $x = \cos^{-1}y$ are implicit functions.

函數一定是方程式，方程式不一定是函數。

Theorem

Rolle's Theorem

Let f be a function that satisfies the following hypotheses :
1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).
3. $f(a) = f(b)$

$\Rightarrow$ Then there is a number c in (a,b) such that $f'(c) = 0$.

Rolle's Theorem 可以拿來證均值定理！

Mean Value Theorem

Let f be a function that satisfies the following hypotheses :
1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).

$\Rightarrow$ Then there is a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

均值定理可以拿來證之後拉格朗的定理！

Corollary

If $f'(x) = 0$ for all x in an interval (a,b), then f is constant on (a,b).

If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f - g$ is constant on (a,b); that is, $f(x) = g(x) + c$ where c is a constant.

反函數微分

微分公式

Let f be differentiable and f^{-1} exists. If $y_0 = f^{-1}(x_0)$ and $f'(y_0) \neq 0$, then f^{-1} is also differentiable at $x = x_0$ and

$$\frac{df^{-1}}{dx}(x_0) = \frac{1}{f'(y_0)} = \frac{1}{\frac{df}{dy}(y_0)}$$

補充 $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Logarithmic Punction

導數 Let $y = \log_b x$.
Then $b^y = x$.

$$(b^y\ln b)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{b^y\ln b} = \frac{1}{x\ln b}$$

你也可以直接代公式。

The Number e as a Limit

1. If $b = e \Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$
2. $\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \forall x \neq 0.$
3. $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)} \quad \forall x \in I$ with $f(x) \neq 0$.

由於我們知道 if $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$，
我們可以把 e 定義為一個極限的值。

$$e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0}e^{\ln(1+x)^{\frac{1}{x}}}$$
$$= \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{n\to\infty}(1+\frac{1}{n})^n \approx 2.7182818$$

What is the number e? (I).pdf（我了解要多聞於自然指數的定義，可以點擊右側連結開啟講義）

Logarithmic Differentiation

步驟
1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

範例1 Differentiate $y = f(x)^{g(x)}$,

$$\ln y = \ln f(x)^{g(x)} = g(x)\ln f(x)$$
$$\frac{y'}{y} = g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}$$

Differentiate $y = x^{\sqrt{x}}$,

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$
$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$
$$y' = y(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}})$$
$$= x^{\sqrt{x}}(\frac{2+\ln x}{2\sqrt{x}})$$

範例2 Differentiate $y = \frac{x\sqrt[3]{x^2+1}}{(3x+2)^5}$,

$$\ln|g(x)| = \frac{3}{4}\ln|x| + \frac{1}{2}\ln|x^2+1| - 5\ln|3x+2|$$
$$\frac{\frac{d}{dx}g(x)}{g(x)} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2} \quad \text{(Differential implicitly with respect to x)}$$
$$\frac{d}{dx}g(x) = g(x)(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2})$$
$$= \frac{x\sqrt[3]{x^2+1}}{(3x+2)^5}(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2})$$

範例3 Calculate the derivative of $y = x(x-1)(x-2)(x-3)$.

$$\ln|y| = \ln|x| + \ln|x-1| + \ln|x-2| + \ln|x-3|$$
$$\frac{y'}{y} = \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right)$$
$$y' = x(x-1)(x-2)(x-3)\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right)$$

The result is valid at all numbers x other than 0, 1, 2, 3.
These are the numbers where $g(x) = 0$.

That logarithmic differentiation fails at the points where $y = 0$ is not a serious deficiency because at these points we can easily apply the product rule. See the following example.

Calculate the derivative of $g(x) = x(x-1)(x-2)(x-3)$ at $x = 3$.

$$g'(x) = \left((x-3)\frac{d}{dx}[x(x-1)(x-2)]\right) + \left(1[x(x-1)(x-2)]\right)$$
$$\Rightarrow g'(3) = 3(3-1)(3-2) = 6.$$

Inverse Trignometric Punction

微分步驟

消反 Let $y = \sin^{-1}x \Rightarrow \sin y = x$

畫圖

微分 $(\sin y)' = x' \Rightarrow \cos y\cdot y' = 1 \Rightarrow y' = \sec y$

還原 $(\sin^{-1}x)' = y' = \sec y = \frac{1}{\sqrt{1-x^2}}$

你也可以直接代公式。

導數

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \forall |x| < 1;$ $\quad$ $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, \forall |x| < 1;$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \forall x \in \mathbb{R};$ $\quad$ $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}, \forall x \in \mathbb{R};$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \forall |x| > 1;$ $\quad$ $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}, \forall |x| > 1.$

Hyperbolic Functions

Hyperbolic Sine and Cosine

Definition

$$\sinh x = \frac{e^x - e^x}{2}$$
$$\cosh x = \frac{e^x + e^x}{2}$$

Derivative

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$

Identities

$\sinh(-x) = -\sinh x$ $\quad$ $\cosh^2 x - \sinh^2 x = 1$

$\cosh(-x) = \cosh x$ $\quad$ $\sinh 2x = 2\sinh x\cosh x$

$\sinh(x+y) = \sinh x\ \cosh y + \cosh x\ \sinh y$ $\quad$ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh(x+y) = \cosh x\ \cosh y + \sinh x\ \sinh y$

Relation to the Hyperbola $x^2 - y^2 = 1$

For each real x, $\cosh^2 x - \sinh^2 x = 1$, and thus the point $(\cosh x, \sinh x)$ lies on the hyperbola.

For each $x > 0$, the number $\frac{1}{2}x$ gives the area of the hyperbolic sector generated by the hyperbolic arc that begins at $(1,0)$ and ends at $(\cosh x, \sinh x)$.

The Other Hyperbolic Punctions

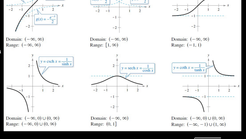
Definition

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$
$$\text{sech}\ x = \frac{1}{\cosh x} \qquad \text{csch}\ x = \frac{1}{\sinh x}$$

Derivative

$$\frac{d}{dx}(\tanh x) = \text{sech}^2\ x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2\ x$$
$$\frac{d}{dx}(\text{sech}\ x) = -\text{sech}\ x\ \tanh x \qquad \frac{d}{dx}(\text{csch}\ x) = -\text{csch}x\ \coth x$$

Gaprh



Inverse Hyperbolic Punction

Theorem

$$\sinh^{-1}x = \ln(x + \sqrt{x^2+1}) \quad x \in \mathbb{R}$$
$$\cosh^{-1}x = \ln(x + \sqrt{x^2-1}) \quad x \geq 1$$
$$\tanh^{-1}x = \frac{1}{2}\ln(\frac{1+x}{1-x}) \quad -1 < x < 1$$
$$\text{csch}^{-1}x = \ln(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}) \quad x \neq 0$$
$$\text{sech}^{-1}x = \ln(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}) \quad 0 < x \leq 1$$
$$\coth^{-1}x = \frac{1}{2}\ln(\frac{x+1}{x-1}) \quad x < -1 \text{ or } x > 1$$

Salas 的書有寫這些函數是怎麼建構出來的，所以才會寫 Theorem，而不是 Definition！

導數

$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$ $\quad$ $\frac{d}{dx}(\text{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$

$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$ $\quad$ $\frac{d}{dx}(\text{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$

$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$ $\quad$ $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$

可以透過前面介紹的反函數微分公式或是直接微分得到！

單變數純量函數導數（應用篇）

算近似值

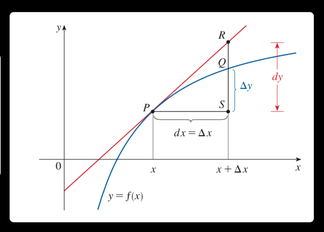
Linear Approximation

之前有提到可微則存在切線，我們可以用切線作線性近似，說明如下。

If f is differentiable at $x = a$,
then the approximating function $L(x) = f(a) + f'(a)(x - a)$ is the linearization of f at a.
The approximation $f(x) \approx L(x)$ is called the linear approximation of f at a.

Differential

The idea behind linear approximations are sometimes formulated in the terminology and notation of differentials.
If $y = f(x)$, where f is a differentiable, then the differential dx is an independent variable;
that is, dx can be given the value of any real number.
The differential dy is then defined in terms of dx by the equation $dy = f'(x)dx$
so dy is an dependent variable; it depends on the values of x and dx.

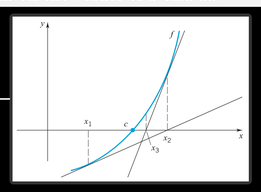


Newton's Method

The method of locating a root of an equation $f(x) = 0$ is call the Newton-Raphson method.

概念

c is the solution of the equation $f(x) = 0$. We can approximate c as follows:
Start at x_1. The tangent line at $(x_1, f(x_1))$ intersects the x-axis at a point x_2 which is closer to c than x_1. The tangent line at $(x_2, f(x_2))$ intersects the x-axis at point x_3 which in turn is closer to c than x_2. In this manner, we obtain numbers $x_1, \ldots, x_n, x_{n+1}$, which more and more closely approximate c.



作法

Suppose that f is twice differentiable.
1. Make an initial estimate $x_1 \in (a, b)$ that is close to c. ($f(x)f''(x) > 0$ for all x between c and x_1.)
2. Determine a new approximation using the iterative relation: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.
3. When $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation.

算不定型極限

Consider $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$:

Indeterminate Forms（不定型）

1.若$x \to a$時，有$f(x) \to 0$及$g(x) \to 0$，則我們稱該極限為「$\dfrac{0}{0}$不定型」.

2.若$x \to a$時，有$f(x) \to \infty$及$g(x) \to \infty$，則我們稱該極限為「$\dfrac{\infty}{\infty}$不定型」.(∞可改為$-\infty$)

上述所提及的$x \to a$可改為$x \to a^+, x \to a^-, x \to \infty$或$x \to -\infty$

L'Hospital's Rule（算不定型極限定理）

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).
若$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$屬於$\dfrac{0}{0}$或$\dfrac{\infty}{\infty}$不定型，則$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ if the limit on the right side exists (or is ∞ or $-\infty$).

其他不定型

Indeterminate Products

Consider $\lim\limits_{x \to a} f(x)g(x)$:
若$x \to a$時，有$f(x) \to 0$及$g(x) \to \infty$，則我們稱該極限為「$0 \cdot \infty$不定型」.(∞可改為$-\infty$)

這種不定型有兩種計算方式，哪個好用就用哪個！

1.因為$\lim\limits_{x \to a} f(x)g(x) = \lim\limits_{x \to a} \dfrac{f(x)}{\frac{1}{g(x)}}$，所以$0 \cdot \infty$不定型可化為$\dfrac{0}{0}$不定型。

2.因為$\lim\limits_{x \to a} f(x)g(x) = \lim\limits_{x \to a} \dfrac{g(x)}{\frac{1}{f(x)}}$，所以$0 \cdot \infty$不定型可化為$\dfrac{\infty}{\infty}$不定型。

Indeterminate Differences

Consider $\lim\limits_{x \to a}[f(x) - g(x)]$:
若$x \to a$時，有$f(x) \to \infty$及$g(x) \to \infty$，則我們稱該極限為「$\infty - \infty$不定型」.

此種不定型的計算重點在於把減改成除，再用$\dfrac{0}{0}$或$\dfrac{\infty}{\infty}$不定型來算。

Indeterminate Powers

Consider $\lim\limits_{x \to a}[f(x)]^{g(x)}$，其中$f(x) > 0$:
(1)若$x \to a$時，有$f(x) \to 0$及$g(x) \to 0$，則我們稱該極限為「0^0不定型」。
(2)若$x \to a$時，有$f(x) \to \infty$及$g(x) \to 0$，則我們稱該極限為「∞^0不定型」。
(3)若$x \to a$時，有$f(x) \to 1$及$g(x) \to \infty$，則我們稱該極限為「1^∞不定型」。

此種不定型的計算重點在於「令$y = f(x)^{g(x)}$」，此時$\lim\limits_{x \to a} \ln y = \lim\limits_{x \to a} g(x) \ln f(x)$ 為 $0 \cdot \infty$ 不定型，然後因為$\lim\limits_{x \to a} f(x)^{g(x)} = \lim\limits_{x \to a} y = \lim\limits_{x \to a} e^{\ln y} = e^{\lim_{x \to a} \ln y}$。因此只要能算出$\lim\limits_{x \to a} \ln y$，就能得到$\lim\limits_{x \to a} f(x)^{g(x)}$的答案。

找極值

極值介紹

Let c be a number in the domain D of a function f. Then $f(c)$ is the

絕對極值
- (Absolute) Maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- (Absolute) Minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

The number $f(c)$ is a

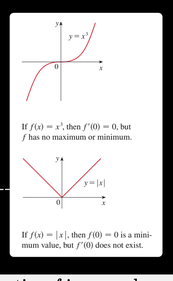
局部極值
- Local Maximum value of f if $f(c) \geq f(x)$ when x is near c.
- Local Minimum value of f if $f(c) \leq f(x)$ when x is near c.

相對極值

Fermat's Theorem

If f has a local maximum or minimum at c and $f'(c)$ exists, then f'(c)=0.

⚠謹記：若 P 則 Q；非 Q 則非 P！
1.就算f'(c)=0，也有可能沒有局部極值！
2.就算f'(c)不存在，也有可能有局部極值！



綜合上述，我們發現極值不是在 $f'(c) = 0$ 就是在 $f'(c)$ 不存在的點上，因此，可以把 Fermat's Theorem 改成如下：
If f has a local maximum or minimum at c,
then c is a critical number of f.

Critical Number

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ doesn't exists.

Increasing/Decreasing

定義

Let f be defined on an interval I.
1. f is increasing on I if for all $x_1, x_2 \in I, x_1 < x_2$ implies $f(x_1) < f(x_2)$.
2. f is decreasing on I if for all $x_1, x_2 \in I, x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function that is increasing or decreasing on an interval is said to be monotonic on the interval.

判別方式

Let f be continuous on a closed interval $[a, b]$ and f be differentiable on an open interval (a, b).
1.If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
2.If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.
3.If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

The First Derivative Test（找相對極值方法 1）

Suppose that c is a critical number of a continuous fumction f.
1.If f' changes from positive to negative at c, then f has a local maximum at c.
2.If f' changes from negative to positive at c, then f has a local minimum at c.
3.If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local extrema at c.

前面的定理只告訴我們若該點有相對極值，
該點一定是 Critical Number 上。
因此我們需要 The First Derivative Test，
來告訴我們在 Critical Number 是否有相對極值！

Concave Upward/Downward

定義

Let f be differentiable on an open interval I.
1.The graph of f is concave upward on I if f' is increasing on I.
2.The graph of f is concave downward on I if f' is decreasing on I.

判別方式

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Inflection Point

Let f be a differentiable on an open interval containing c.
The point $(c, f(c))$ is called an inflection point of the graph of f
if the concavity of f changes from upward to downward or downward to upward at this point.

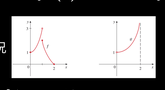
The Second Derivative Test（找相對極值方法 2）

Suppose that f'' is continuous near c.
1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
3. If $f'(c) = 0$ and $f''(c) = 0$ or doesn't exist, then the test fails.

通常會用 The First Derivative Test！

絕對極值

「有可能」沒有絕對極值的情況



還記得我們在連續函數提到的 Extreme Value Theorem嗎？這邊提醒大家
只要極值定理中的其中一個條件沒有被滿足就「有可能」沒有絕對極值。
1.左邊的圖沒有連續（只有絕對極小）
2.右邊的圖沒有在閉區間上（沒有絕對極大也沒有絕對極小）

The Closed Interval Method（找絕對極值方法 1）

If f is continuous on a closed interval$[a, b]$,
we can follow below to find the absolute maximum and minimum :
1.Find critical numbers of f in (a, b).
2.Evaluate f at each critical points in (a, b).
3.Evaluate f at the end-points in $[a, b]$.
4.Compare the values of 2 and 3.（這些值中最大的就是絕對極大值；最小的就是絕對極小值！）

The First Derivative Test for Absolute Extrema（找絕對極值方法 2）

Suppose that c is a critical number of a continuous function f defined on an interval.
1.If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
2.If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

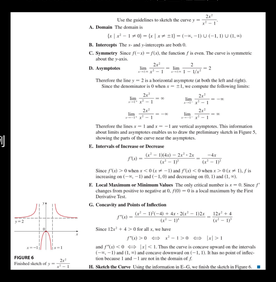
除了這兩種方法，
也可以在找到全部的相對極值後，
直接和端點比較就知道最大最小值了！

Curve Skeching

步驟

- Domain : 確定定義域
- Intercepts : 找出 x 截距、y 截距
- Symmetry： 確定函數是否具有對稱性
- Asymptotes : 判斷是否有水平、垂直、斜漸近線
- Intervals of Increase or Decrease : 找出函數的遞增遞減區間
- Local Extrema : 利用 Critical Numbers 與一次導數判別法找出局部極值
- Concavity and Points of Inflection : 利用反曲點確定函數的凹性
- Sketch the Curve : 綜合上述畫圖

範例



Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. Domain The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. Intercepts The x- and y-intercepts are both 0.

C. Symmetry Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D. Asymptotes

$$\lim_{x \to \infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote (at both the left and right).
Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

E. Intervals of Increase or Decrease

$$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ ($x \neq -1$) and $f'(x) < 0$ when $x > 0$ ($x \neq 1$), f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. Local Maximum or Minimum Values The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G. Concavity and Points of Inflection

$$f''(x) = \frac{(x^2 - 1)^2(-4) + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection because 1 and -1 are not in the domain of f.

H. Sketch the Curve Using the information in E–G, we finish the sketch in Figure 6.

FIGURE 6
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

單變數純量函數積分（概念篇）

Definite Integral

先備知識：Sigma

Proposition

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Sums of Powers

$$\sum_{i=1}^{n} c = cn \quad \text{if } c \text{ is a constant}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

前言

Let $f\colon [a,b] \to \mathbb{R}$ be a non-negative continuous function, and R be the region enclosed by the graph of the function f, the x-axis and straight lines $x=a$ and $x=b$. We consider computing $A(R)$, the area of R.

計算曲線下的面積——第一步（考慮非負連續函數）

Partition $[a,b]$ into n sub-intervals with equal length, and let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$.

By the Extreme Value Theorem, for each $1 \le i \le n$, f attains its maximum and minimum on $[x_{i-1}, x_i]$;

thus for $1 \le i \le n$, there exist $M_i, m_i \in [x_{i-1}, x_i]$ such that $\begin{cases} f(M_i) = \text{the maximum of } f \text{ on } [x_{i-1}, x_i] \\ f(m_i) = \text{the minimum of } f \text{ on } [x_{i-1}, x_i] \end{cases}$

$\begin{cases} S(n) \equiv \sum_{i=1}^{n} f(M_i)\Delta x \text{ is called the upper sum of } f \\ s(n) \equiv \sum_{i=1}^{n} f(m_i)\Delta x \text{ is called the lower sum of } f \end{cases}$ for the partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$,

計算曲線下的面積——第二步（考慮非負連續函數）

By the definition of the upper sum and lower sum, we find that for each $n \in \mathbb{N}$, $\sum_{i=1}^{n} f(m_i)\Delta x \le A(R) \le \sum_{i=1}^{n} f(M_i)\Delta x$.

If the limits of the both sides exist and are identical as Δx approaches 0 (which is the same as n approaches infinity), by the Squeeze Theorem we conclude that $A(R)$ is the same as the limit.

Riemann Integrals

Partition

A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is said to be a partition of the closed interval $[a,b]$ if $a = x_0 < x_1 < \cdots < x_n = b$.

Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n = b\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ is the number $\max\{x_1 - x_{i-1} \mid 1 \le i \le n\}$; that is, $\|\mathcal{P}\| \equiv \max\{x_i - x_{i-1} \mid 1 \le i \le n\}$. A partition is called regular if $x_i - x_{i-1} = \|\mathcal{P}\|$ for all $1 \le i \le n$.

Riemann Sums

Let $f\colon [a,b] \to \mathbb{R}$ be a function. A Riemann sum of f for the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ is a sum which takes the form $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$, where the set $S = \{c_0, c_1, \cdots, c_{n-1}\}$ satisfies that $x_{i-1} \le c_i \le x_i$ for each $1 \le i \le n$.

$c_i \in [x_{i-1}, x_i]$是可以任選的，且其被稱為 Sample Points。

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a,b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A is unique and is called the Riemann integral of f on $[a,b]$ and is denoted by $\displaystyle\int_{[a,b]} f(x)\,dx$.

Riemann Integral 存在判準一：

連續必可積

If f is continuous on $[a,b]$, or f has only a finite number of jump discontinuous, then f is Riemann integrable on $[a,b]$.

有界必有積

Let $f\colon [a,b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a,b]$, then f is bounded on $[a,b]$; that is, there exists $M > 0$ such that $|f(x)| \le M$ whenever $x \in [a,b]$.

事實上，“f is continuous on $[a,b]$，意即”f 在 $[a,b]$ 上是連續函數，所以下面兩件事是要注意的：

(1) 函數 f 並未規定間斷點要怎麼一回事，對 f 來一定可積一回事：for all $x \in [a,b]$,

$$f(x) = \begin{cases} 0 & , \text{if } x \in \mathbb{Q}, \\ 1 & , \text{if } x \notin \mathbb{Q}. \end{cases}$$

這樣的 f 不可積。

(2) 函數 f 並未規定間斷點可以有無限點不連續（這樣的不連續專指 jump discontinuous），對 f 仍可積一回事：

$$f(x) = \begin{cases} 1 & , \text{if } 0 < x < 2, \\ 2 & , \text{if } 2 < x \le 3, \end{cases}$$

這樣的 f 是可積的。

定義

For conventional reason, the Riemann integral of f over the interval with left-end point a and right-end point b is written as $\displaystyle\int_a^b f(x)\,dx$, and is called the definite integral of f from a to b. We also note that here in the representation of the integral, x is a dummy variable; that is, we can use any symbol to denote the independent variable; thus $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$.

$\displaystyle\int$: Integral Sign $\quad f(x)$: Integrand $\quad a$: Lower Limit $\quad b$: Upper Limit

範例

曲線下的面積（連續非負函數）

Let $f\colon [a,b] \to \mathbb{R}$ be a non-negative and continuous function. The area of the region enclosed by the graph of f, the axis, and the vertical lines $x=a$ and $x=b$ is $\displaystyle\int_a^b f(x)\,dx$.

非負連續函數與有正有負的連續函數在定積分計算下的差異

(1) $\displaystyle\int_a^b f(x)\,d$ can be interpreted as the area if $f(x) \ge 0$ for all $x \in [a,b]$.

(2) $\displaystyle\int_a^b f(x)\,d$ can be interpreted as the net area if $f(x) \in \mathbb{R}$（f 有正有負） for all $x \in [a,b]$.

EXAMPLE Evaluate $\int_0^1 \sqrt{x^3 - x}\,dx$.

SOLUTION We use Theorem 6.6 "Write $f(x) = \sqrt{x^3 - x}$ for $x \in \mathbb{R}$ or $x \ge 0$, and $g_{n,k} = \frac{k}{2^n}$, $x_{n,k} = \frac{k-1}{2^n}$, $k \ge 1$."

Then the sequence of the subdivision $\{x_0 = 0, x_1 = \frac{1}{2^n}, \cdots, x_{2^n} = 1\}$, $x_k = k \cdot 2(2n) = 2(2n \cdot k)$, $x_k = 0 \cdot 2(2n) = 0$, and in general:

$$x_k = k \cdot \left(\frac{1}{2}\right) = \frac{k}{2}$$

Thus

$$\begin{aligned} \int_0^1 \sqrt{x^3 - x}\,dx &= \lim_{n\to\infty} \sum_{k=1}^{2^n} f(x_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(\frac{1}{2}\right)\frac{1}{2^n} \\ &= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{2^n} \left(\frac{1}{2^n}\right)^3 \cdot 4^n \left(\frac{1}{2}\right) \\ &= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{2^n} \left(\frac{1}{2^n}\right)^3 \cdot \frac{2^{2n}}{2} \\ &= \lim_{n\to\infty} \left[\frac{1}{2} \sum_{k=1}^{2^n} \frac{1}{2^n}\right] \cdot \frac{2^{2n}}{2} \\ &= \lim_{n\to\infty} \left[\frac{1}{2} \left(\frac{1}{2^n} \cdot 2^n\right)\right] \cdot \frac{2^{2n}}{2} \\ &= \lim_{n\to\infty} \left[\frac{1}{2} \left(\frac{1}{2^n} \cdot \frac{1}{2^n} \cdot 2^n\right)\right] \cdot \frac{2^{2n}}{2} \\ &= \frac{1}{2} \cdot 1 \cdot \frac{1}{2} \cdot 2^{2n} = 2^{2n} \end{aligned}$$

Proposition

Integral Rules Suppose that f is continuous on $[a,b]$.

$$\int_a^b c\,dx = c(b-a) \quad \text{where } c \text{ is a constant}$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{where } c \text{ is a constant}$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx \quad (a < c < b)$$

Comparison Properties

If $f(x) \ge 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx \ge 0$.

If $f(x) \ge g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

If $m \le f(x) \le M$ for all $x \in [a,b]$, then $m(b-a) \le \displaystyle\int_a^b f(x)\,dx \le M(b-a)$

Corollary

Suppose f is continuous on $[-a,a]$.

1. If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
2. If f is even $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Definite Integral as a Punction of Upper Limit

Prerequirement

Until now we have integrated only from left to right: from a number a to a number b greater than a. We integrate in the other direction by defining $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

The integral from any number to itself is defined to be zero: $\displaystyle\int_c^c f(x)\,dx = 0$ if f is defined at $x = c$

With this additional conventions, the additivity condition $\displaystyle\int_a^c f(x)\,dx + \int_c^d f(x)\,dx = \int_a^d f(x)\,dx$ holds for all choices of a, b, c from an interval on which f is continuous, no matter what the order of a, b, c happens to be.

Definition

If f is continuous on $[a,b]$ and c is any number in $[a,b]$, then for each x in $[a,b]$, the integral is a number, and consequently we can define a function F on $[a,b]$ by setting $F(x) = \displaystyle\int_c^x f(t)\,dt$.

Theorem

Let f be continuous on $[a,b]$ and let c be any number in $[a,b]$. The function F defined on $[a,b]$ by setting $F(x) = \displaystyle\int_c^x f(t)\,dt$ is continuous on $[a,b]$, differentiable on (a,b), and has derivative $F'(x) = f(x)$ for all x in (a,b).

The Fundamental Theorem of Calculus 微積分基本定理

定理內容

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is an antiderivative of f, that is, a function F such that $F' = f$.

Antiderivative：A function F is called an antiderivative on an interval I if $F'(x) = f(x)$ for all x in I.

概念補充 Suppose that g, h is a differentiable function of x with values in $[a,b]$.

1. $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ and then $\dfrac{d}{dx}\displaystyle\int_x^a f(t)\,dt = -\frac{d}{dx}\int_a^x f(t) = -f(x)$
2. $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$
3. $\dfrac{d}{dx}\displaystyle\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$

Mean Value Theorem for Integral

First

If f is continuous on $[a,b]$, then there is at least one number $c \in (a,b)$ for which $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$.

This number $f(c)$ is called the average value of f on $[a,b]$.

Second

If f and g are continuous on $[a,b]$, and g is nonnegative, then there is a number $c \in (a,b)$ for which $\displaystyle\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

This number $f(c)$ is called the g-weighted average of f on $[a,b]$.

Indefinite Intergral

Definition

We begin with a continuous function f. If F is an antiderivative for f on [a,b], then $\displaystyle\int_a^b f(x)\,dx = \big[F(x)\big]_a^b$.

If C is a constant, then $\big[F(x) + C\big]_a^b = \big[F(b) + C\big] - \big[F(a) + C\big] = F(b) - F(a) = \big[F(x)\big]_a^b$.

Thus we can write $\displaystyle\int_a^b f(x)\,dx = \big[F(x) + C\big]_a^b$. If we have no particular interest in the interval $[a,b]$ but wish instead to emphasize that F is an antiderivative for f, which on open intervals simply means $F' = f$, then we omit the a and b and simply write $\displaystyle\int f(x)\,dx = F(x) + C$. Antiderivative expressed in this manner are called indefinite integrals.

不定積分表

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int a\,dx = ax + C$	
$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$ (or $r \neq -1$)	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \frac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
$\int \cosh x\,dx = \sinh x + C$	$\int \sinh x\,dx = \cosh x + C$

一些積分技巧

透過微積分基本定理和一些微分法則，我們可以有這些積分方法！

Substitution 變數置換

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

不定積分版

$$\int f(g(x))g'(x)\,dx \xrightarrow{du = g'(x)dx} \int f(u)\,du \xrightarrow{F' = f} F(u) + C$$

定積分版

If $g'(x)$ is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx \xrightarrow{du = g'(x)dx} \int_{g(a)}^{g(b)} f(u)\,du \xrightarrow{F' = f} [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

變數置換是從 Chain Rule 來的，
說明如下：

$$\int f(g(x))g'(x)\,dx \xrightarrow{F' = f} \int F'(g(x))g'(x)\,dx \xrightarrow{\text{by the Chain Rule}} \int \frac{d}{dx}[F(g(x))]\,dx \xrightarrow{\text{by 微積分基本定理}} F(g(x)) + C.$$

⚠ 其實左邊的寫法只是為了方便計算，並不嚴謹，因為積分符號的 dx 並不是 Differential，正確的寫法應該是從 Chain Rule 去看。
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx \xrightarrow{F' = f} \int F'(g(x))g'(x)\,dx \xrightarrow{\text{by the Chain Rule}} \int \frac{d}{dx}[F(g(x))]\,dx \xrightarrow{\text{by 微積分基本定理}} F(g(x)) + C.$$

範例

Find $\displaystyle\int 2x\sqrt{1+x^2}$.

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx$$

不定積分

$$\begin{aligned} &= \int \sqrt{u}\,du \qquad \text{Let } u = 1 + x^2 \text{ then } du = 2xdx \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C \end{aligned}$$

範例

Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\,dx$ $\begin{cases} \text{法一是先代入新變數，算完之後再變回舊變數。} \\ \text{法二則是先代入新變數後，把積分範圍換成新變數的積分範圍。} \end{cases}$

定積分

法一：$\displaystyle\int_0^4 \sqrt{2x+1} = \int \sqrt{2x+1}\Big|_{x=0}^{x=4}$ Let $u = 2x+1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$.

$$\begin{aligned} &= \int \sqrt{u} \cdot \frac{1}{2}\,du\Big|_{x=0}^{x=4} = \frac{1}{2}\int u^{\frac{1}{2}}\,du\Big|_{x=0}^{x=4} \\ &= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C\Big|_{x=0}^{x=4} = \frac{1}{3}u^{\frac{3}{2}} + C\Big|_{x=0}^{x=4} \qquad u = 2x+1 \\ &= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C\Big|_{x=0}^{x=4} = \frac{26}{3} \end{aligned}$$

法二：Let $u = 2x+1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$. When $x = 0, u = 2(0)+1 = 1$ When $x = 4, u = 2(4)+1 = 9$

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2} \cdot \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=9} = \frac{26}{3}$$

Integration by Parts 分部積分

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

不定積分版 If we let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.

We can write $\displaystyle\int u\,dv = uv - \int v\,du$.

分部積分是從Product Rule for Differentiation 來的，
說明如下：

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\Rightarrow f(x)g(x) = \int [f(x)g'(x) + g(x)f'(x)]\,dx$$

$$\Rightarrow f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

定積分版

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

單變數純量函數積分（技巧篇）

Trigonometric Substitution 三角代換

- $\sqrt{(a^2-x^2)^m},\ m\in\mathbb{N}$：令 $x=a\sin\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$，則 $\sqrt{a^2-x^2}=a\cos x$
- $\sqrt{(a^2+x^2)^m},\ m\in\mathbb{N}$：令 $x=a\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}$，則 $\sqrt{a^2+x^2}=a\sec x$
- $\sqrt{(x^2-a^2)^m},\ m\in\mathbb{N}$：令 $x=a\sec\theta, 0\le\theta<\frac{\pi}{2}$ 或 $\pi\le\theta<\frac{3\pi}{2}$，則 $\sqrt{x^2-a^2}=a\tan x$

這個方法其實是一種 Inverse Substitution，和前面學的變數變換的差別是
變數變換是令 $u=f(x)$，也就是新變數是舊變數的函數；
三角代換則是令 $x=f(\theta)$，也就是舊變數是新變數的函數（須為一對一函數，所以限制 θ 範圍）！

Tangent Half-Angle Substitution 半角置換法

Let $t=\tan\frac{x}{2}$, then we have that

1. $\sin x=\frac{2t}{1+t^2}$
2. $\cos x=\frac{1-t^2}{1+t^2}$
3. $dx=\frac{2}{1+t^2}\,dt$

適用於被積分函數含有 Sin 或 Cos 之有理函數者！

範例

Find $\int\frac{1}{\sin x+2\cos x+3}\,dx$

$$\int\frac{1}{\frac{2t}{1+t^2}+2\cdot\frac{1-t^2}{1+t^2}+3}\cdot\frac{2}{1+t^2}\,dt=2\int\frac{1}{t^2+2t+5}\,dt$$

$$=2\int\frac{1}{2^2+(t+1)^2}d(t+1)\qquad \text{Let } t+1=2\tan\theta, \text{ then } d(t+1)=2\sec^2\theta\,d\theta$$

$$=2\int\frac{1}{4(1+\tan^2\theta)}2\sec^2\theta\,d\theta\qquad 1+\tan^2\theta=\sec^2\theta$$

$$=\int\frac{1}{\sec^2\theta}\sec^2\theta\,d\theta\quad=\theta\qquad \tan\theta=\frac{t+1}{2}\Rightarrow\theta=\tan^{-1}(\frac{t+1}{2})$$

$$=\tan^{-1}(\frac{t+1}{2})+C\qquad t=\tan\frac{x}{2}$$

$$=\tan^{-1}(\frac{1+\tan\frac{x}{2}}{2})+C$$

Partial Fractions 部分分式

Rational functions $\frac{P(x)}{Q(x)}$ 有兩種：

先備知識

If $\deg(P)\ge\deg(Q)$, then $\frac{P(x)}{Q(x)}$ is called "improper". （例如 $\frac{x^2+1}{x-3}$）

If $\deg(P)<\deg(Q)$, then $\frac{P(x)}{Q(x)}$ is called "proper". （例如 $\frac{x-3}{x^2+1}$）

方法說明

定理告訴我們，任何 Proper Rational Fucntion 都能被表示成 $\frac{A}{(ax+b)^k}$ 或 $\frac{Ax+B}{(ax^2+bx+c)^k}$ 形式的部份分式的和，
其中 ax^2+bx+c 必須 "irreducible"（意思是，ax^2+bx+c 的判別式 b^2-4ac 必須 <0。詳細說明如下：

1. 分母 $Q(x)$ 因式分解成 $(ax+b)^k$ 或 $(ax^2+bx+c)^k$ 的形式。
2. 每個 $(ax+b)^k$ 會對應到 $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$，分子都是常數。
3. 每個 $(ax^2+bx+c)^k$ 會對應到 $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$，分子都是一次式。

《注意》遇到 Improper 就使用「長除法」把它改成 Polynomial + Proper 再算。例如：$\frac{x^3+x}{x-1}=x^2+x+2+\frac{2}{x-1}$.

情況

- Case 1
  The denominator $Q(x)$ is a product of a distinct linear factors.
  例如：$Q(x)=2x^3+3x^2-2x=x(2x-1)(x+2)$
- Case 2
  The denominator $Q(x)$ is a product of a distinct linear factors, some of which are repeated.
  例如：$Q(x)=x^3-x^2-x+1=(x-1)^2(x+1)$
- Case 3
  The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
  例如：$Q(x)=(x-2)(x^2+1)(x^2+4)$
  ⚠ 通常會用到這個公式，請熟記：$\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}(\frac{x}{a})+c$
- Case 4
  The denominator $Q(x)$ contains a repeated irreducible quadratic factor.
  例如：$Q(x)=x(x-1)(x^2+1)^3$

每個 Case 的做法不太一樣，請看 Stewart 原文書範例。

Rationalizing Substitutions

我們可以透過適當的變數變換將無理函數變成有理函數，尤其是當遇到 $\sqrt[n]{g(x)}$，讓 $u=\sqrt[n]{g(x)}$ 通常有效。

例：Evaluate $\int\frac{\sqrt{x+4}}{x}\,dx$

Let $u=\sqrt{x+4}$. Then $u^2=x+4$, so $x=u^2-4$ and $dx=2u\,du$.

$$\int\frac{\sqrt{x+4}}{x}\,dx=\int\frac{u}{u^2-4}2u\,du=2\int\frac{u^2}{u^2-4}\,du=2\int(1+\frac{4}{u^2-4})\,du$$

$$=2\int du+8\int\frac{1}{u^2-4}\,du=2\int du+8\cdot\frac{1}{2\cdot2}\int(\frac{1}{u-2}-\frac{1}{u+2})\,du$$

$$=2u+8\cdot\frac{1}{2\cdot2}\ln|\frac{u-2}{u+2}|+c$$

$$=2\sqrt{x+4}+2\ln|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}|+c$$

Trigonometric Integral

三角函數積分

一次

(1) $\int\sin x\,dx=-\cos x+c$ (2) $\int\cos x\,dx=\sin x+c$

(3) $\int\tan x\,dx=\int\frac{\sin x}{\cos x}\,dx=\int\frac{-1}{\cos x}\,d\cos x=-\ln|\cos x|+c=\ln|\sec x|+c$

(4) $\int\cot x\,dx=\int\frac{\cos x}{\sin x}\,dx=\int\frac{1}{\sin x}\,d\sin x=\ln|\sin x|+c$

(5) $\int\sec x\,dx=\int\sec x\frac{\sec x+\tan x}{\sec x+\tan x}\,dx=\int\frac{1}{\sec x+\tan x}\,d(\sec x+\tan x)=\ln|\sec x+\tan x|+c$

(6) $\int\csc x\,dx=\int\csc x\frac{\csc x-\cot x}{\csc x-\cot x}\,dx=\int\frac{1}{\csc x-\cot x}\,d(\csc x-\cot x)=\ln|\csc x-\cot x|+c$

二次

(1) $\int\sin^2x\,dx=\int\frac{1-\cos 2x}{2}\,dx=\int\frac{1}{2}\,dx-\frac{1}{4}\int\cos 2x\,d\,2x=\frac{x}{2}-\frac{1}{4}\sin 2x+c$

(2) $\int\cos^2x\,dx=\int\frac{1+\cos 2x}{2}\,dx=\int\frac{1}{2}\,dx+\frac{1}{4}\int\cos 2x\,d\,2x=\frac{x}{2}+\frac{1}{4}\sin 2x+c$

(3) $\int\tan^2x\,dx=\int(\sec^2x-1)\,dx=\int\sec^2x\,dx-\int dx=\tan x-x+c$

(4) $\int\cot^2x\,dx=\int(\csc^2x-1)\,dx=\int\csc^2x\,dx-\int dx=-\cot x-x+c$

(5) $\int\sec^2x\,dx=\tan x+c$ (6) $\int\csc^2x\,dx=-\cot x+c$

三次

(1) $\int\sin^3x\,dx=\int\sin^2x(\sin x\,dx)=-\int(1-\cos^2x)\,d\cos x=\int(-1)\,d\cos x+\int\cos^2x\,d\cos x=-\cos x+\int t^2\,dt=-\cos x+\frac{t^3}{3}+c\quad t=\cos x$

(2) $\int\cos^3x\,dx=\int\cos^2x(\cos x\,dx)=\int(1-\sin^2x)\,d\sin x=\sin x-\frac{\sin^3x}{3}+c$

(3) $\int\tan^3x\,dx=\int\frac{\tan^2x}{\sec x}(\sec x\tan x\,dx)=\int\frac{\sec^2x-1}{\sec x}\,d(\sec x)=\int(\sec x-\frac{1}{\sec x})\,d(\sec x)=\frac{1}{2}(\sec x)^2-\ln|\sec x|+c$

(4) $\int\cot^3x\,dx=\int\frac{\cot^2x}{\csc x}(\csc x\cot x\,dx)=-\int\frac{\csc^2x-1}{\csc x}\,d\csc x=\int\frac{1}{\csc x}\,d\csc x-\int\csc x\,d\csc x=\ln|\csc x|-\frac{1}{2}\csc^2x+c$

(5) $\int\sec^3x\,dx=\int\sec x(\sec^2x\,dx)=\int\sec x\,d(\tan x)$ (Integration by Part)

$$=\sec x\tan x-\int\tan x\,d(\sec x)=\sec x\tan x-\int(\sec^2x-1)\sec x\,dx$$

$\Rightarrow$ 故 $\int\sec^3x\,dx=\sec x\tan x-\int\sec^3x\,dx+\int\sec x\,dx$

$\Rightarrow\int\sec^3x\,dx=\frac{1}{2}(\sec x\tan x+\int\sec x\,dx)=\frac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|+c)$

(6) $\int\csc^3x\,dx=\int\csc x(\csc^2x\,dx)=-\int\csc x\,d\cot x=-\csc x\cot x-\int\cot^2\csc x\,dx$

$$=-\csc x\cot x-\int(\csc^2x-1)\csc x\,dx=-\csc x\cot x-\int\csc^3x\,dx+\int\csc x\,dx$$

$\Rightarrow$ 故 $\int\csc^3x\,dx=\frac{1}{2}(-\csc x\cot x+\int\csc x\,dx)$

$\Rightarrow\int\csc^3x\,dx=\frac{1}{2}(-\csc x\cot x+\ln|\csc x-\cot x|+c)$

三次以上的可以使用這些公式！
如果我們的 Integrand 是函數的次方，我們可以用之前學過的分部積分幫我們降低次數，因此我們稱之為 Reduction Formula！

(1) $\int\sin^nx\,dx=-\frac{\sin^{n-1}\cos x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\,dx$

(2) $\int\cos^nx\,dx=\frac{\sin x}{n}\cos^{n-1}x+\frac{n-1}{n}\int\cos^{n-2}x\,dx$

(3) $\int\tan^nx\,dx=\frac{1}{n-1}\tan^{n-1}x-\int\tan^{n-2}x\,dx$

(4) $\int\cot^nx\,dx=-\frac{\cot^{n-1}x}{n-1}-\int\cot^{n-2}x\,dx$

(5) $\int\sec^nx\,dx=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}x\,dx$

(6) $\int\csc^nx=-\frac{\csc^{n-2}x\cot x}{n-1}+\frac{n-2}{n-1}\int\csc^{n-2}x\,dx$

高次三角函數積分

$\int\sin^mx\cos^nx$ 型 $m,n>0$

1. m 為奇數，設法都換成 $\cos x$：

$$\int\sin^mx\cos^nx\,dx=\int\sin^{2k+1}x\cos^nx\,dx=\int(\sin^2x)^k\cos^nx\sin x\,dx$$

$$=-\int(1-\cos^2x)^k\cos^nx\,d(\cos x)$$

2. n 為奇數，設法都換成 $\sin x$：

$$\int\sin^mx\cos^nx\,dx=\int\sin^mx\cos^{2k+1}x\,dx=\int\sin^mx(\cos^2x)^k\cos x\,dx$$

$$=-\int\sin^mx(1-\sin^2x)^k\,d(\sin x)$$

3. m n 全為偶 $\Rightarrow$ 利用 $\sin x\cos x=\frac{\sin 2x}{2}$ or 降次公式 $\begin{cases}\sin^2x=\frac{1-\cos 2x}{2}\\ \cos^2x=\frac{1+\cos 2x}{2}\end{cases}$.

$\int\tan^mx\sec^nx$ 型 $m,n>0$

1. n 為偶數

$$\int\tan^mx\sec^nx\,dx=\int\tan^mx\sec^{2k}x\,dx=\int\tan^mx(\sec^2x)^{k-1}\sec^2\,dx$$

$$=\int\tan^mx(\tan^2x+1)^{k-1}\,d(\tan x)$$

2. m 為奇數

$$\int\tan^m\sec^n\,dx=\int(\tan^2x)^k\sec^{n-1}\sec x\tan x\,dx=\int(\sec^2x-1)^k\sec^{n-1}x\,d(\sec x).$$

3. m 為偶數，n 為奇數

$$\int\tan^mx\sec^nx\,dx=\int\tan^{2k}\sec^nx\,dx=\int(\tan^2x)^k\sec^n\,dx=\int(\sec^2x-1)^k\sec^nx\,dx$$

乘開後 $\Rightarrow$ 用 $\int\sec^nx\,dx=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}x\,dx\quad(n\ge2)$

$$\int\sec x\,dx=\ln|\sec x+\tan x|+c\,;\int\sec^2x\,dx=\tan x+c$$

倍次三角函數積分

- 求 $\int\sin mx\cos nx\,dx$：利用 $2\sin A\cos B=\sin(A-B)+\sin(A+B)$
- 求 $\int\sin mx\sin nx\,dx$：利用 $2\sin A\sin B=\cos(A-B)-\cos(A+B)$
- 求 $\int\cos mx\cos nx\,dx$：利用 $2\cos A\cos B=\cos(A-B)+\cos(A+B)$

單變數純量函數瑕積分

定義

積分區間無限或是積分函數無限的定積分就叫做瑕積分。
接下來就是要介紹如果遇到瑕積分該怎麼做的技巧。

我們之前學過，如果定積分的積分範圍是 $[a,b]$，則函數可積的條件至少是「在積分範圍內有界」或是「函數在積分範圍內只有有限多個 jump 型的不連續點」，但瑕積分所考慮的積分區間有可能是 $[a,\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$，函數也可能會在積分範圍內跑到正負無窮大 （不是 jump 型的不連續），所以「瑕積分的積分值不一定存在」。如果瑕積分的積分值存在，我們就稱此瑕積分收斂 (Convergent)，如果它的積分值不存在，我們就稱此瑕積分發散 (Divergent)。請注意，「瑕積分的積分值不存在」指的是計算出來的結果振盪（即值不唯一）或是計算出來的結果跑到正負無窮大。

種類

Infinite Intervals

1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ provided the limit exists.

2. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$ provided the limit exists.

3. If both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \quad \forall\, a \in \mathbb{R}$$

常見錯誤說明

1. 自創寫法：
$$\int_0^\infty x^3\,dx = \frac{1}{4}x^4\Big|_0^\infty = \infty \quad \text{雖然答案對，但寫法不對！}$$
2. 自創定義：
$$\int_{-\infty}^\infty x\,dx = \lim_{t\to\infty}\int_{-t}^t x\,dx = \lim_{t\to\infty}\frac{x^2}{2}\Big|_{-t}^t = \lim_{t\to\infty}\left(\frac{t^2}{2} - \frac{(-t)^2}{2}\right) = 0 \quad \text{寫法不對！}$$
3. 自創性質：
$$f(x) \text{ is an odd function} \Rightarrow \int_{-\infty}^\infty x\,dx = 0 \quad \text{寫法不對！其實} \int_{-\infty}^\infty x\,dx \text{ 發散！}$$

總而言之，就是一定要照定義算，不要自己幻想一些奇怪的性質或公式甚至是靠直覺！

Proposition

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx \quad \forall\, a < c$, provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall\, c < b$, provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

Discontinuous Integrands

1. If f is Riemann integrable on $[a,c]$ $\forall\, a < c < b$ and has an infinite discontinuity at b, then $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$ provided the limit exists.

2. If f is Riemann integrable on $[c,b]$ $\forall\, a < c < b$ and has an infinite discontinuity at a, then $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$ provided the limit exists.

3. If f has an infinite discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

常見錯誤說明

1. 看到積分範圍有限就沒有檢查是不是瑕積分：
$$\int_{-1}^3 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{-1}^3 = \frac{-4}{3} \quad \text{寫法不對！其實} \int_{-1}^3 \frac{1}{x^2}\,dx \text{ 發散！}$$
2. 自創寫法：
$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = 2\sqrt{x-2}\Big|_2^5 = 2\sqrt{3} - 2\sqrt{0} = 2\sqrt{3} \quad \text{寫法不對！}$$
請試著算算看，正確答案雖然還是 $2\sqrt{3}$，但這只是剛好，一定要使用定義來計算！

從現在開始看到積分就要注意它是不是瑕積分，而不是看到題目就積分。

Proposition

If f is Riemann integrable on $[a,c]$ $\forall\, a < c < b$, and f has an infinite discontinuity at a or b, then $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall a < c < b$, provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

Infinite Intervals & Discontinuous Integrands

例：$\int_{-\infty}^\infty \frac{1}{x^2}\,dx$

把它想辦法拆解成前面兩種的其中一種，就可以算了！

Theorem

Direct Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

P.S. Similar result also holds for other kinds of improper integrals.

Limit Comparison Test

Let f and g be positive continuous functions on the interval $[a,\infty)$.
If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$ for some $0 < L < \infty$, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^\infty \mathrm{g(x)\,dx}$ converges.

P.S. Similar result also holds for other kinds of improper integrals.

有時候沒辦法找到瑕積分的值，但知道它是收斂還是發散很重要！

常見瑕積分

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{發散} & p \leq 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{發散} & p \geq 1 \\ \frac{1}{1-p} & p < 1 \end{cases}$$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \begin{cases} \text{發散} & p \leq 1 \\ \frac{1}{(p-1)^2} & p > 1 \end{cases}$$

$$\int_0^1 \frac{\ln x}{x^p}\,dx = \begin{cases} \text{發散} & p \geq 1 \\ \frac{-1}{(1-p)^2} & p < 1 \end{cases}$$

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \begin{cases} \text{發散} & p \leq 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

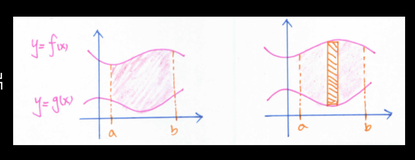
單變數純量函數積分（應用篇）

Areas between Curves

Integrating with Respect to x

Case 1 $f(x) \geq g(x)$ for all $x \in [a, b]$

- 計算方法 $A = \lim_{n\to\infty} \sum_{i=1}^{n} \left\{ f(x_i^*) - g(x_i^*) \right\} \Delta x_i = \int_a^b \left[f(x) - g(x) \right] dx.$ 如下圖所示
- 圖片

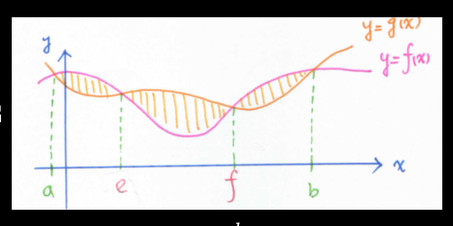


Case 2 Sometimes $f(x) \geq g(x)$ and sometimes $f(x) \leq g(x)$

- 計算方法

$$A = \int_a^b \left| f(x) - g(x) \right| dx. \quad \text{如下圖所示}$$
$$= \int_a^e \left[f(x) - g(x) \right] dx + \int_e^f \left[g(x) - f(x) \right] dx + \int_f^b \left[f(x) - g(x) \right] dx.$$

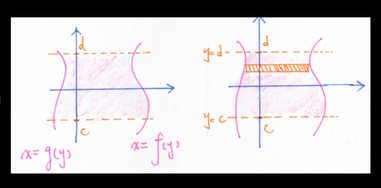
- 圖片



Integrating with Respect to y

Case 1 $f(y) \geq g(y)$ for all $y \in [c, d]$

- 計算方法 $A = \lim_{n\to\infty} \sum_{i=1}^{n} \left\{ f(x_i^*) - g(x_i^*) \right\} \Delta y_i = \int_c^d \left[f(y) - g(y) \right] dy.$ 如下圖所示
- 圖片

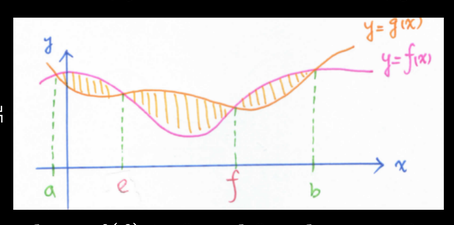


Case 2 Sometimes $f(y) \geq g(y)$ and sometimes $f(y) \leq g(y)$

- 計算方法

$$A = \int_c^d \left| f(y) - g(y) \right| dy. \quad \text{如下圖所示}$$
$$= \int_c^e \left[f(y) - g(y) \right] dy + \int_e^f \left[g(y) - f(y) \right] dy + \int_f^d \left[f(y) - g(y) \right] dy.$$

- 圖片



Area in Polar Coordinates

Case 1

Let R be the region bounded by $r = f(\theta)$, $\theta = a$ and $\theta = b$, where $f(\theta) \geq 0$ and $0 < b - a \leq 2\pi$.

Then the area A of R is $A = \int_a^b \frac{1}{2} \left[f(\theta) \right]^2 d\theta.$

Case 2

Let R be the region bounded by $r = f(\theta)$, $r = g(\theta)$, $\theta = a$ and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$.

Then the area A of R is $A = \int_a^b \frac{1}{2} \left[f(\theta) \right]^2 d\theta - \int_a^b \frac{1}{2} \left[g(\theta) \right]^2 d\theta$.

Volumes

The Disk Method 圓盤法

非旋轉型

- 計算方式

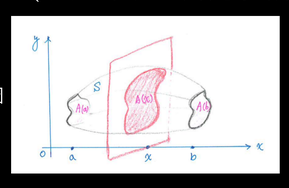
如果切開的截面和 x 軸垂直，我們可以用公式 $V = \int_a^b A(x)\, dx.$

如果切開的截面和 y 軸垂直，我們可以用公式 $V = \int_c^d A(y)\, dy.$

(下面的示意圖是以截面和 x 軸垂直為例！)

注意：立體物體的截面要怎麼切，都是看你方便計算而決定！

- 示意圖



旋轉型

無中空型

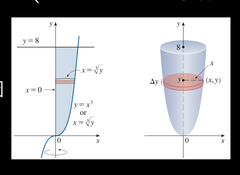
- 計算方式

如果是繞 x 軸旋轉，則旋轉體體積為 $V = \int_a^b A(x)\, dx = \int_a^b \pi [f(x)]^2\, dx$.

如果是繞 y 軸旋轉，則旋轉體體積為 $V = \int_c^d A(y)\, dy = \int_c^d \pi [g(y)]^2\, dy$.

(下面的示意圖是以繞 y 軸為例！)

- 示意圖



中空型

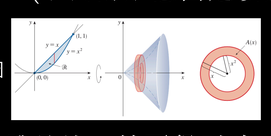
- 計算方式

若旋轉體是兩函數繞 x 軸而得，則體積公式為：$V = \int_a^b \pi [R(x)]^2\, dx - \int_a^b \pi [r(x)]^2\, dx = \pi \int_a^b \left\{ [R(x)]^2 - [r(x)]^2 \right\} dx$

若旋轉體是兩函數繞 y 軸而得，則體積公式為：$V = \int_c^d \pi [R(y)]^2\, dy - \int_c^d \pi [r(y)]^2\, dy = \pi \int_c^d \left\{ [R(y)]^2 - [r(y)]^2 \right\} dy$

(下面的示意圖是以繞 x 軸為例！)

- 示意圖



The Shell Method 剝殼法

- 計算方式

1.《當旋轉體是繞 y 軸而得到時》由上往下看，把旋轉體由上往下切，切成一圈一圈的，對每一圈的面積做積分！

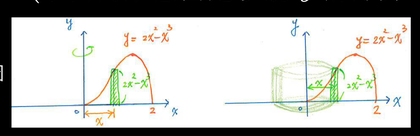
此時旋轉體體積的公式為：$V = \int_a^b 2\pi x \cdot f(x)\, dx.$ 公式裡的 $2\pi x$ 代表的是圓周長，$f(x)$ 對應的是高度，dx 對應的是厚度。

2.《當旋轉體是繞 x 軸而得到時》由左往右看，把旋轉體由左往右切，切成一層一層的，對每一層的面積做積分！

此時旋轉體體積的公式為：$V = \int_c^d 2\pi y \cdot f(y)\, dy.$ 公式裡的 $2\pi y$ 代表的是圓周長，$f(y)$ 對應的是高度，dy 對應的是厚度。

(下面的示意圖是以繞 y 軸為例！)

- 示意圖



Arc Length

- 弧長公式

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$,

$$\text{is} \quad L = \int_a^b \sqrt{1 + \left[f'(x) \right]^2}\, dx$$

If g' is continuous on $[c, d]$, then the length of the curve $x = g(y)$, $c \leq x \leq d$,

$$\text{is} \quad L = \int_c^d \sqrt{1 + \left[g'(y) \right]^2}\, dy$$

- 弧長函數 $s(x) = \int_a^x \sqrt{1 + \left[f'(t) \right]^2}\, dt$

弧長函數可以幫助我們計算從一個特定起始點到任意結束點的弧長！

Area of a Surface of Revolution

公式

繞 x 軸旋轉

Let S denote the area of a surface of revolution.

Assume f is positive and continuously differentiable on $[a, b]$. If the curve is $y = f(x)$, $a \leq x \leq b$ and rotates about $x - axis$, then

$$S = \int_a^b 2\pi f(x) \sqrt{1 + \left[f'(x) \right]^2}\, dx \quad \text{（請參考下列左圖）}$$

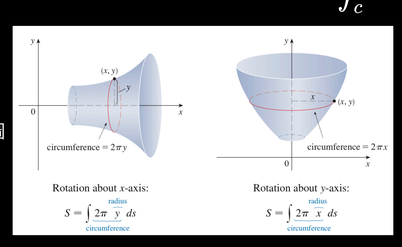
繞 y 軸旋轉

Let S denote the area of a surface of revolution.

Assume g is positive and continuously differentiable on $[c, d]$. If the curve is $x = g(y)$, $c \leq y \leq d$ and rotates about $y - axis$, then

$$S = \int_c^d 2\pi g(y) \sqrt{1 + \left[g'(y) \right]^2}\, dy \quad \text{（請參考下列右圖）}$$

示意圖



參數方程式

Parametric Equations

Parametric equations are equation such that x and y are both functions of a third variable t (call a parameter):

$$x = f(t),\ y = g(t),\ t \in [a, b].$$

Parametric Curve: Each t determines a point (x, y). As t varies, $(x, y) = (f(t), g(t))$ varies and traces out a curve, which we call a parametric curve.

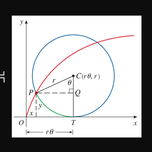
Cycloid 擺線

定義

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.

參數化過程

圖片



說明

We choose the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin) as parameter. From the figure, we can know that $|OT| = \text{arc } PT = r\theta$. Therefore the center of the circle is $C(r\theta, r)$. Let coordinates of P be (x, y).

Then from the figure we see that $\begin{cases} x = |OT| - |PQ| = r\theta - r\sin\theta \\ y = |TC| - |QC| = r - r\cos\theta \end{cases}$. Therefore parametric equations of the cycloid are $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$ $\theta \in \mathbb{R}$.

Calculus with Parametric Curves

Tangents

切線之斜率計算

Suppose f and g are differentiable functions and we want to find the slope of a tangent line at a point on the parametric curve $x = f(t),\ y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

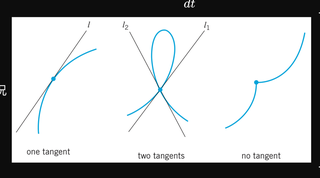
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

更直觀來看的話，可以把 $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ 想成是切向量的 y 分量除以 x 分量！

為了瞭解這個參數方程曲線的凹性，我們有時也需要計算二階導數 $\frac{d^2y}{dx^2}$：

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

平面曲線的切線可能情況



判斷是否有切線的方法

(1)若 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 不同時為0，則必存在一條切線！若 $\begin{cases} \frac{dy}{dt} = 0，則C有水平切線。 \\ \frac{dx}{dt} = 0，則C有垂直切線。 \end{cases}$

(2)若 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 同時為0，什麼情形都有可能發生，必須另外找方法判斷。

兩者皆為零之說明範例

Verify that the following examples satisfy the equation $f'(0) = g'(0) = 0$ and that the given description holds at the point where $t = 0$.

1. $f(t) = t^3,\ g(t) = t^2$; cusp.
2. $f(t) = t^3,\ g(t) = t^5$; horizontal tangent.
3. $f(t) = t^5,\ g(t) = t^3$; vertical tangent.
4. $f(t) = t^3 - 1,\ g(t) = 2t^3$; tangent with slope 2.
5. $f(t) = t^2,\ g(t) = t^2 + 1$; no tangent line.

Area Under Parametric Curves

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, if $F(x) \geq 0$.

If the curve can be traced out by the parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases},\ \alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

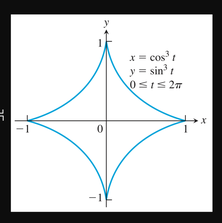
計算方式

Let $y = F(x)$ be corresponding Cartesian equation. Suppose $a \leq x \leq b$ corresponds to $\alpha \leq t \leq \beta$.

(1) If $t = \alpha$ derives leftmost endpoint, then
$$A = \int_\alpha^\beta g(t)f'(t)\,dt\,.$$

(2) If $t = \beta$ derives leftmost endpoint, then
$$A = \int_\beta^\alpha g(t)f'(t)\,dt\,.$$

範例 Find the area enclosed by the astroid.



做法

$$\begin{aligned} A &= 4\int_0^1 y\,dx && \text{4 times area under } y \text{ from } x = 0 \text{ to } x = 1 \\ &= 4\int_0^{\frac{\pi}{2}} (\sin^3 t)(3\cos^2 t \sin t)\,dt && \text{Substitution for } y \text{ and } dx \\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt && \sin^4 t = \left(\frac{1 - \cos 2t}{2}\right)^2 \\ &= \frac{3\pi}{8} \end{aligned}$$

Cartesian Coordinates

Arc Length

計算方式

If curve C is given by $x = f(t),\ y = g(t),\ \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once, then length of C is $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

範例 Calculate the circumference of the unit circle.

作法一

Parametrizing the unit circle by setting $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases},\ t \in [0, 2\pi]$, we have $\begin{cases} x'(t) = -\sin t \\ y'(t) = \cos t \end{cases}$

and the arc length formula gives $L(C) = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} 1\,dt = 2\pi$.

作法二

Parametrizing the unit circle by setting $\begin{cases} x(t) = \cos 2t \\ y(t) = \sin 2t \end{cases},\ t \in [0, 2\pi]$, we have $\begin{cases} x'(t) = -2\sin 2t \\ y'(t) = 2\cos 2t \end{cases}$

and the arc length formula gives $L(C) = \int_0^{2\pi} \sqrt{4\sin^2 2t + 4\cos^2 2t}\,dt = \int_0^{2\pi} 2\,dt = 4\pi$.

很明顯，第一個作法算出來的周長才是對的。為什麼第二個作法會錯呢？
因為這個公式算出來的會是實際經過的長度，第一個作法只有畫一圈，但是
第二個作法畫了兩圈，因此得出兩倍周長！我們之前有說過參數化曲線的方式
不只一種，你可以用其他參數方程表示同一條曲線，但若要用參數方程算曲線長度，
請記得這個參數方程只能經過曲線上的每個點一次！

和定積分弧長公式比較

Suppose that C is the graph of a continuously differentiable function $y = (x),\ x \in [a, b]$.

We can parametrize C by setting $\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases}$ $t \in [a, b]$. Since $x'(t) = 1$ and $y'(t) = f'(t)$, gives $L(C) = \int_a^b \sqrt{1 + [f'(t)]^2}\,dt$. Replacing t by x, we can write: $L(C) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$.

這個式子跟我們前面用積分算 Arc Length 的式子是一樣的，這兩個方法的差異在於前面教的公式
只能用在 C is the graph of a continuously differentiable function $y = (x)$ 的情況，但後者則沒有這個限制！

Arc Length Function

For a parametric curve C given by $x = f(t),\ y = g(t)$, where f' and g' are continuous, we let $s(t)$ be the arc length along C from an initial point $(f(\alpha), g(\alpha))$ to a point $(f(t), g(t))$ on C. By formula, the arc length function s for parametric curves is $s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du$

Surface Area

If curve is given by $x = f(t), y = g(t), a \leq t \leq b$ and is rotated about $x-$axis, where f', g' are continuous and $g(t) \geq 0$, then resulting surface area S is

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\,.$$

If curve is given by $x = f(t), y = g(t), a \leq t \leq b$ and is rotated about $y-$axis, where f', g' are continuous and $f(t) \geq 0$, then resulting surface area S is

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\,.$$

和定積分公式比較

我們以繞 x 軸為例。Suppose that C is the graph of a continuously differentiable nonnegative function $y = f(x),\ x \in [a, b]$.

We can parametrize C by setting $\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases}$ $t \in [a, b]$. Since $x'(t) = 1$ and $y'(t) = f'(t)$, the area of surface generated by revolving C about the x axis is given by the formula $A = \int_a^b 2\pi y \sqrt{1 + [f'(t)]^2}\,dt$. Replacing t by x, we can write: $A = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\,dx$.

這個式子跟我們前面用積分算 Surface Area 的式子是一樣的，這兩個方法的差異在於前面教的公式
只能用在 C is the graph of a continuously differentiable nonnegative function $y = (x)$ 的情況，但後者則沒有這個限制！

Polar Coordinates

Tangents

If a curve C is given by $r = f(\theta)$, we can write parametric equations for a polar curve $r = f(\theta)$ as follows

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta \text{（轉換成直角坐標下的參數方程式）}$$

根據參數方程式的切線斜率公式，我們可以得到 $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

判斷是否有切線的方法

(1)若 $\frac{dx}{d\theta}$ 和 $\frac{dy}{d\theta}$ 不同時為0，則必存在一條切線！若 $\begin{cases} \frac{dy}{d\theta} = 0，則C有水平切線。 \\ \frac{dx}{d\theta} = 0，則C有垂直切線。 \end{cases}$

(2)若 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 同時為0，什麼情形都有可能發生，必須另外找方法判斷。

Area

Let R denote the region bounded by $r = f(\theta),\ \theta = a$ and $\theta = b$, where $f(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. Then the area A of R is

$$A = \int_a^b \frac{1}{2}(f(\theta))^2\,d\theta = \int_a^b \frac{1}{2}r^2\,d\theta$$

Arc Length

If a curve C is given by $r = f(\theta)$, we can write parametric equations for a polar curve $r = f(\theta)$ as follows

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta \text{（轉換成直角坐標下的參數方程式）}$$

因為 $\begin{cases} \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{cases} \xrightarrow{\text{Using } \cos^2\theta + \sin^2\theta = 1} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

所以，我們可以把弧長公式 $L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$ 寫成 $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\,.$

總而言之

A curve with polar equation $r = f(\theta),\ a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\,.$$

數列與級數

Infinite Sequence

- **Definition** A function $f: \mathbb{N} \to \mathbb{R}$ is called a sequence.
- **Limit of a Sequence**
  - A sequence $\{a_n\}$ has the limit L and we write
    $$\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L \ as \ n \to \infty$$
    if for every $\varepsilon > 0$ there is a corresponding interger N such that
    $$\text{if} \quad n > N \quad then \quad |a_n - L| < \varepsilon$$
  - A sequence $\begin{cases} \text{that has a limit is said to be convergent.} \\ \text{that has no limit is said to be divergent.} \end{cases}$

  若極限值存在則唯一
- **Infinite Limit** The notation $\lim_{n\to\infty} a_n = \infty$ means that for every $M > 0$ there is an interger N such that $\text{if} \quad n > N \quad then \quad a_n > M$
- **Proposition**
  - If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = L$.

    L'Hospital Rule不能用在數列的極限上，但可以用在函數的極限上，再引用得到數列的極限。
  - **Limit Laws for Sequences** Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant.
    - $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
    - $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$
    - $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$
    - $\lim_{n\to\infty} {a_n}^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$
    - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$
  - If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.
  - **Squeeze Theorem** If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
- **Monotonic and Bounded Sequence**
  - **Monotonic 定義** A sequence is called monotonic if it is either increasing or decreasing.
  - **Bounded 定義** If a sequence is bounded above or below, then it is called a bounded sequence.
  - **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

    也就是 $\begin{cases} \text{有上界且遞增} \\ \text{有下界且遞減} \end{cases}$ 數列必收斂。

Infinite Series

- **Partial Sum** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum :
  $$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$
  If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s \in \mathbb{R}$,
- **級數的斂散性**
  - **Convergent** then the series is called convergent and we write $\sum_{n=1}^{\infty} a_n = s$ (The sum of the series)
  - **Divergent??????** If the sequence $\{s_n\}$ is divergent, then the series is called divergent.
- **Example**
  - **Geometric Series** If $|r| \ge 1$, the geometric series is divergent.
    If $|r| < 1$, the geometric series is convergent and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
  - **Harmonic Series**
- **Proposition**
  - If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
  - **無窮級數運算** Suppose that $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant.
    - $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
    - $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$
- **正項級數**
  - **Integral Test**
    - If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
    - If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

    Remainder Estimate
    If $R_n = s - s_n$ then
    $$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$
    and we can get
    $$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx$$
  - **特殊例子**
    - **Geometric Series** If $|r| \ge 1$, the geometric series is divergent.
      If $|r| < 1$, the geometric series is convergent and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
    - **p-series (From Integral Test)** The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.
  - **Comparison Tests**
    - **Direct Comparison Test (收斂的話可以更進一步預估和)**
      - If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.
      - If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.
    - **Limit Comparison Test** If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$, then either both series converge or both diverge.
- **交錯級數審斂法**
  If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots \ (b_n > 0)$
  satisfies the conditions $\begin{matrix} b_{n+1} \le b_n & \text{for all } n \\ \lim_{n\to\infty} b_n = 0 & \end{matrix}$ then the series is convergent.

  Remainder Estimate
  If $s = \sum(-1)^{n-1}b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies
  $$b_{n+1} \le b_n \quad \text{and} \quad \lim_{n\to\infty} b_n = 0$$
  then
  $$|R_n| = |s - s_n| \le b_{n+1}$$
- **斂散性判斷法（沒有變數的級數）**
  - **一般級數（拿到先加絕對值）**
    - 若級數加絕對值後收斂，則原級數收斂（此時稱原級數絕對收斂）

      絕對收斂級數重排後，和不變！
    - 若級數加絕對值後發散，則必須另尋他法測試原級數
      - 原級數收斂（此時稱原級數條件收斂）

        條件收斂級數可以透過重排改變和！
      - 原級數發散
  - **Ratio Test**
    - If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series is absolutely convergent.
    - If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then the series is divergent.
    - If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the Ratio Test is inconclusive.
  - **Root Test**
    - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series is absolutely convergent.
    - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series is divergent.
    - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

    如果 Ratio Test 結果等於 1，那就不需要試Root Test，結果會等於 1！（反之亦然）

  Test for Divergence（起手式）
  If $\lim_{n\to\infty} a_n$ doesn't exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.
- **冪級數 (Power Series，簡寫 P.S.)：帶有變數 x，次數無窮大的多項式。** $\sum_{n=0}^{\infty} a_n(x-c)^n$

  P.S.的斂散性會與 x 代入的值有關，使P.S.收斂的 x 值會形成 x 的收斂區間而收斂區間的中點必為 P.S. 的中心 c。
  - **收斂區間與半徑**
    - **求法**
      - **Ratio Test（最常用）**
      - **Root Test**

      收斂區間端點需另外判斷
    - **定理** For a power series, there are only three possibilities :
      1. The series converges only when $x = a$.
      2. The series converges for all x.
      3. There is a number $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.
  - **Geometric Series** $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$
  - **Power Series 微分和積分定理**
    If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$,
    then the function f defined by $f(x) = c_0 + c_1(x-a) + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$
    is differentiable on the interval $(a - R, a + R)$ and
    1. $f'(x) = c_1 + 2c_2(x-a) + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
    2. $\int f(x)\,dx = C + co(x-a) + c_1\frac{(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

    The radii of convergence of the P.S. in 1 and 2 are both R.

    將函數轉為級數是為了：
    1.積分沒有 Elementary Antiderivatives 的函數！
    2.Approximating Functions by Polynomials.

    透過幾何級數和定理我們可以得到其他函數的級數！
- **Taylor and Maclaurin Series**
  - **定理** If f has a power series representation at a, that is, if
    $$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x - a| < R$$
    then its coefficients are given by the formula
    $$c_n = \frac{f^{(n)}(a)}{n!}$$
  - **定理** If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if $\lim_{n\to\infty} R_n(x) = 0$ for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.
  - **Taylor's Inequality** If $|f^{(n+1)}(x)| \le M$ for $|x - a| \le d$ then the remainder $R_n(x)$ of the Taylor series satisfies the inequality
    $$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x - a| \le d$$
  - **Taylor's Series of Important Functions（端點要額外確認喔！）**
    - $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$
    - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$
    - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$
    - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all } x$
    - $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)} \quad |x| < 1$
    - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad |x| < 1$
    - $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty}\binom{k}{n}x^n \quad |x| < 1 \text{ and } k \in \mathbb{R}$

    泰勒級數可以拿來算特殊極限！此外，我們也可以透過操作這些已知的級數來得到新的泰勒級數！

    $$\text{Taylor}: \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$
    $$\text{Maclaurin}: \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$
    ⚠Maclaurin Series 是泰勒級數的特例！

第一個定理告訴我們只要函數可以用冪級數表示，那它就一定可以寫成泰勒級數。
第二個定理則幫助我們確定函數可不可以用冪級數表示。
第三個定理則是幫助我們去證明定理二中的 $\lim_{n\to\infty} R_n = 0$
⚠ 在使用定理二、三時常用到以下極限（請背起來）：
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every } x \in \mathbb{R}$$

單變數向量函數（概念篇）

之所以要探討 Coordinate System 是因為微積分基本上都在討論

這四種函數 $\begin{cases} \text{Scalar Function of Single Variable } f\colon \mathbb{R} \to \mathbb{R} \\ \text{Vector Function of Single Variable } \mathbf{f}\colon \mathbb{R} \to \mathbb{R}^n \\ \text{Scalar Function of Several Variables } f\colon \mathbb{R}^n \to \mathbb{R} \\ \text{Vector Function of Several Variables } \mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m \end{cases}$ where $n, m \in \mathbb{N}$

n-dimensional Coordinate System $\mathbb{R}^n$

- 1-dimensional coordinate system $\mathbb{R}$ (即數線)
  - 選一個固定點，將其記為原點，過原點畫一條直線，選一邊為正，以某長度為單位長，則數線上每一個點可以在 $\mathbb{R}$ 中找到某一個數且僅有一個數與之對應，反之亦然。也就是，數線和實數的對應是 one-to-one 且 onto。因此，數線 $\cong \mathbb{R}$（數線會等價於實數），數線上的點對應到的數稱為此點的坐標。
- 2-dimensional coordinate system $\mathbb{R}^2$
  - 過某條數線的原點，畫垂直該數線的直線，以原單位長為單位長，並選取一方向為正，而得一條新的數線，原數線稱為 x 軸，新數線稱為 y 軸，則此平面稱為 $x-y$ plane。將平面上的點分別垂直投影到 x 軸和 y 軸，則在 $x-y$ plane 上的每一個點可找到一個數對且僅有一個數對 (x, y) 與其對應，反之亦然。故 $x-y$ plane $\cong \mathbb{R}^2$，此數對 (x, y) 稱為此點的坐標。
  - 補充：數對的建立：Let A and B be two sets. We define a new set $A \times B$ by the following way: $A \times B = \{(a, b) \mid a \in A, b \in B\}$. And, we construct $\mathbb{R}^2$ by the following way: $\mathbb{R} \times \mathbb{R} \triangleq \mathbb{R}^2$. 依此類推可建立 $\mathbb{R}^n$。
- 3-dimensional coordinate system $\mathbb{R}^3$
  - 過 $x-y$ plane 的原點作一垂直該平面的直線，以原單位長為單位長，並選取一方向為正，並稱此新數線為 z 軸。將空間中的點分別垂直投影到 x 軸和 y 軸和 z 軸，則在此 $x-y-z$ space 的每一點可找到一個數對且僅有一個數對 (x, y, z) 與其對應，反之亦然。故 $x-y-z$ space $\cong \mathbb{R}^3$，此數對 (x, y) 稱為此點的坐標。
- 4-dimensional coordinate system $\mathbb{R}^4$
  - 過 $x-y-z$ space 的原點作一垂直 $x-y-z$ space 的直線，以原單位長為單位長，選一方向為正，並稱此數線為 t 軸，則得一個 4 維的空間。在物理上，稱之為時空。Similarly, $x-y-z-t$ space $\cong \mathbb{R}^4$.
  - 依此類推可得 $n-$dimensional coordinate system $\mathbb{R}^n$。
- The Distance
  - Let $P = (x_1, x_2, \ldots, x_n)$ and $Q = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.
  - The distance $\|PQ\|$ between P and Q is defined by $\|PQ\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

Definition

A function whose domain is a set of real numbers and whose range is a subset of n-space V_n is called a vector function of a real variable.

和 Scalar Function 差異

Functions such as $f(t) = 2 + 3t$, $f(t) = \sin 2t$ assign real numbers to real numbers. They are called scalar functions.
Functions such as $\mathbf{f}(t) = \mathbf{r}_0 + t\mathbf{d}$, $\mathbf{f}(t) = \sin t\,\mathbf{a} + \cos t\,\mathbf{b}$ assign vectors to real numbers. They are called vector functions.

Vector functions can be built up from scalar functions in an obvious manner. From scalar functions f_1, f_2, f_3 that share a common domain, we can construct the vector function $\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$.

Limit

- Definition
  - (1) $\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{L}$ provided that $\lim_{t \to t_0} \|\mathbf{f}(t) - \mathbf{L}\| = 0$.
  - (2) $\lim_{t \to t_0} \mathbf{f}(t) = \left\langle \lim_{t \to t_0} f_1(t), \ldots, \lim_{t \to t_0} f_n(t) \right\rangle$ provided that the limits of the component functions exist.
  - (3) Let I be the domain of $\mathbf{f}$. The notation $\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{L}$ means $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\|\mathbf{f}(t) - \mathbf{L}\| < \varepsilon$ whenever $0 < |t - t_0| < \delta$ and $t \in I$.
  - 每本書寫法不同，但沒差，因為這三個定義是等價的。⚠ 極限存在則唯一
- Proposition — Limit Laws
  - Let $\mathbf{f}$ and $\mathbf{g}$ be vector function and let u be a scalar function.
  - Suppose that $\begin{cases} \lim_{t \to t_0} \mathbf{f}(t) = \mathbf{L} \\ \lim_{t \to t_0} \mathbf{g}(t) = \mathbf{M} \\ \lim_{t \to t_0} u(t) = A. \end{cases}$
    - $\lim_{t \to t_0} \mathbf{f}(t) + \mathbf{g}(t) = \mathbf{L} + \mathbf{M}$
    - $\lim_{t \to t_0} \alpha \mathbf{f}(t) = \alpha \mathbf{L}$
    - $\lim_{t \to t_0} u(t)\mathbf{f}(t) = A\mathbf{L}$
    - $\lim_{t \to t_0} \mathbf{f}(t) \cdot \mathbf{g}(t) = \mathbf{L} \cdot \mathbf{M}$
    - $\lim_{t \to t_0} \mathbf{f}(t) \times \mathbf{g}(t) = \mathbf{L} \times \mathbf{M}$

Continuity

- Definition
  - $\mathbf{f}$ is said to be continuous at t_0 provided that $\begin{cases} \lim_{t \to t_0} \mathbf{f}(t) = \mathbf{f}(t_0). \\ \text{each component of } \mathbf{f} \text{ is continuous at } t_0. \end{cases}$
  - 這兩個定義等價。
- Proposition
  - If $\mathbf{f}(t), \mathbf{g}(t)$ and $u(t)$ are continuous at a and c is a constant, then the following functions are also continuous at a :
  - 1. $\mathbf{f} \pm \mathbf{g}$ 2. $c\mathbf{f}$ 3. $u\mathbf{f}$ 4. $\mathbf{f} \cdot \mathbf{g}$ 5. $\mathbf{f} \times \mathbf{g}$

Derivative

- Definition
  - The vector function $\mathbf{f}$ is said to be differentiable at t provided that $\begin{cases} \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h} \text{ exists.} \\ \text{the derivatives of the component functions exist. } (\mathbf{f}'(t) = \langle \mathbf{f}_1'(t), \ldots, \mathbf{f}_n'(t) \rangle) \end{cases}$
  - If the limit exists, it is called the derivative of $\mathbf{f}$ at t and is denoted by $\mathbf{f}'(t)$.
  - 這兩個定義等價。
- Proposition
  - 可微必連續 If $\mathbf{f}$ is differentiable at t, then $\mathbf{f}$ is continuous at t.
  - Differentiation Rules — Suppose that $\mathbf{f}$ and $\mathbf{g}$ are differentiable vector function, c is a vector, α is a scalar, and $\mathbf{u}$ is a scalar function.
    - $(\mathbf{f} \pm \mathbf{g})'(t) = \mathbf{f}'(t) \pm \mathbf{g}'(t)$
    - $(\alpha \mathbf{f})'(t) = \alpha \mathbf{f}'(t)$
    - $(u\mathbf{f})'(t) = u(t)\mathbf{f}'(t) + u'(t)\mathbf{f}(t)$
    - $(\mathbf{f} \cdot \mathbf{g})'(t) = [\mathbf{f}(t) \cdot \mathbf{g}'(t)] + [\mathbf{f}'(t) \cdot \mathbf{g}(t)]$
    - $(\mathbf{f} \times \mathbf{g})'(t) = [\mathbf{f}(t) \times \mathbf{g}'(t)] + [\mathbf{f}'(t) \times \mathbf{g}(t)]$
    - $(\mathbf{f} \circ u)'(t) = \mathbf{f}'(u(t))u'(t)$
    - $(u\mathbf{c})'(t) = u'(t)\mathbf{c}$
  - If $\mathbf{r}(t)$ is differentiable, then the function $r = \|\mathbf{r}\|$ is differentiable where it is not 0 and $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = r\frac{dr}{dt}$.
  - Corollary — Let $I \subseteq \mathbb{R}$ be an interval, and $\mathbf{r}\colon I \to \mathbb{R}^m$ be a differentiable vector function. If $\|\mathbf{r}(t)\|$ is a constant function on I, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \forall\, t \in I$.
  - If $\mathbf{r}(t)$ is differentiable, then where $r = \|\mathbf{r}\| \neq 0$
    $\frac{d}{dt}(\frac{\mathbf{r}}{r}) = \frac{1}{r^3}[(\mathbf{r} \times \frac{\mathbf{dr}}{\mathbf{dt}}) \times \mathbf{r}]$.

Integral

- Definition
  - (1) A vector function $\mathbf{f}\colon [a, b] \to \mathbb{R}^n$ is said to be integrable on $[a, b]$ if there exists a vector $\mathbf{A}$ such that $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that if $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, any Riemann sum of $\mathbf{r}$ for $\mathcal{P}$ $\left(\sum_{k=1}^{n} \mathbf{r}(\xi_k)(t_k - t_{k-1})\right)$ locates in $(\mathbf{A} - \varepsilon, \mathbf{A} + \varepsilon)$, where the vector $\mathbf{A} \pm \varepsilon$ is the vector obtained by adding or substracting ε from each component of $\mathbf{A}$. The vector $\mathbf{A}$, if exists, is written as $\int_a^b \mathbf{r}(t)\,dt$.
  - (2) $\int_a^b \mathbf{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt, \ldots, \int_a^b f_n(t)\,dt\right)$ provided that the integrals of the component functions exist.
  - 這兩個定義等價。⚠ 定積分存在則唯一
- Proposition
  - 連續必可積 If a vector function $\mathbf{F}$ is continuous on a closed interval $[a, b]$, then $\mathbf{F}$ is integrable on $[a, b]$.
  - Integral Rules
    - $\int_a^b [\mathbf{f}(t) \pm \mathbf{g}(t)]\,dt = \int_a^b \mathbf{f}(t)\,dt \pm \int_a^b \mathbf{g}(t)\,dt$
    - $\int_a^b \alpha \mathbf{f}(\mathbf{t})\,dt = \alpha \int_a^b \mathbf{f}(t)\,dt$ for every constant scalar α.
    - $\int_a^b u(t)\mathbf{c}\,dt = \mathbf{c} \int_a^b u(t)\,dt$ for every constant vector c.
    - $\int_a^b [\mathbf{c} \cdot \mathbf{f}(t)]\,dt = \mathbf{c} \cdot (\int_a^b \mathbf{f}(t)\,dt)$ for every constant vector $\mathbf{c}$
    - $\int_a^b [\mathbf{c} \times \mathbf{f}(t)]\,dt = \mathbf{c} \times (\int_a^b \mathbf{f}(t)\,dt)$ for every constant vector $\mathbf{c}$
    - $\|\int_a^b \mathbf{f}(t)\,dt\| \leq \int_a^b \|\mathbf{f}(t)\|\,dt$
- Theorem
  - Assume $\mathbf{f}(t)$ is continuous on $[a, b]$. If $c \in [a, b]$, define A to be the vector function given by $\mathbf{A}(x) = \int_c^x \mathbf{f}(t)\,dt$ if $a \leq x \leq b$.
  - Then $A'(x)$ exists, and we have $\mathbf{A}'(x) = \mathbf{f}(x)$ for each x in (a, b).
- The Fundamental Theorem of Calculus 微積分基本定理
  - If $\mathbf{F}$ has continuous derivative $\mathbf{F}'$ on an open interval I.
  - Then, for each choice of c and x in I ,we have $\mathbf{F}(x) = \mathbf{F}(c) + \int_c^x \mathbf{F}'(t)\,dt$.

單變數向量函數（應用篇）

Curve

Definition

A subset C in the plane (or space) is called a curve if C is the image of an interval $I \subseteq \mathbb{R}$ under a continuous vector function $\mathbf{r}$.

The continuous function $\mathbf{r}\colon I \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a parametrization of the curve, and the equations $\begin{cases} x = r_1(t) \\ y = r_2(t), \ t \in I \\ z = r_3(t) \end{cases}$（以三維為例），

where x, y, and z are simply scalar functions of t, are called parametric equations of the curve, and t is called a parameter. ⚠ 參數化方式不唯一！

- **Simple**: A curve C is called simple if it has an injective parametrization; that is, there exists $\mathbf{r}\colon I \to \mathbb{R}^3$ such that $\mathbf{r}(I) = C$ and $\mathbf{r}(x) = \mathbf{r}(y)$ implies that $x = y$.
  - 就是這個函數是一對一的意思。
- **Closed**: A curve C with parametrization $\mathbf{r}\colon I \to \mathbb{R}^3$ is called closed if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\mathbf{r}(a) = \mathbf{r}(b)$.
- **Smooth**: A smooth curve is a curve with differentiable parametrization $\mathbf{r}\colon I \to \mathbb{R}^3$ such that $\mathbf{r}'(t) \neq 0$ for all $t \in I$.
  - 就是整個函數都可以微分的意思且微分不為 0。

Arc Length

計算方式

Let C be a curve parametrized by an injective continuously differentiable parametrization $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$.

Then length of C is $L = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$.

- 其實這也蠻直觀的，將曲線上每一個微小的切向量的長度相加，當這些微小切向量的長度之和在取極限時，自然就得到了曲線的長度。這裡的積分本質上就是在處理這樣的極限情況。

The arc length function s for a curve given by a vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \quad a \le t \le b$,

is $s(t) = \int_\alpha^t \|\mathbf{r}'(u)\|\, du = \int_\alpha^t \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2 + (\frac{dz}{du})^2}\, du.$

Arc Length Function

Definition

補充：By the fundamental Theorem of Calculus, $\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}.$

Then by Substitution Rule for Definite Integrals, $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt = \int_a^b ds.$

最後一個式子告訴我們把所有微量長度相加，直觀來看，會得到整條曲線的長度！

弧長參數

Since $\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0$, the function $s(t)$ is a one-to-one increasing function.
Thus, no two points of C can lie at the same arc distance from $\mathbf{r}(a)$. It follows that for each $s \in [0, L]$, there is a unique point $\mathbf{R}(s)$ on C at arc distance s from $\mathbf{r}(a)$.
The vector function $\mathbf{R}(s),\ s \in [0, L]$ parametrizes C by arc length. We call s the arc length parameter for the curve.

補充 1：Since $\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| > 0$ the function $s(t)$ has a differentiable inverse $t(s)$ and we can write $\frac{dt}{ds} = \frac{1}{\|\frac{d\mathbf{r}}{dt}\|}$（套導數單元教過的反函數公式）. Since both $\mathbf{R}(s(t))$ and $\mathbf{r}(t)$ lies at arc distance $s(t)$ from $\mathbf{r}(a)$, $\mathbf{R}(s(t)) = \mathbf{r}(t)$. Therefore $\mathbf{R}(s) = \mathbf{r}(t(s))$. Differentiation gives $\frac{d\mathbf{R}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{d\mathbf{r}}{dt}\frac{1}{\|\frac{d\mathbf{r}}{dt}\|}$.

Taking the norm of both sides, we have $\frac{d\mathbf{R}}{ds} = 1$. Thus, for a curve parametrized by arc length, the tangent vector can change in direction but not in length: the tangent vector maintains length 1.

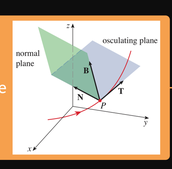
補充 2：Moreover, if the curve $C\colon \mathbf{r}(t),\ t \in [0.b]$ has tangent vector of constant length 1, then the parametrization is by arc length and the length of the curve is b.

- 範例：Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from (1,0,0) in the direction of increasing t.

  $\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{2} \Rightarrow s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t.$

  Therefore $t = \frac{s}{\sqrt{2}}$ and the required reparametrization is obtained by substituting for t: $\mathbf{r}(t(s)) = \cos\frac{s}{\sqrt{2}}\,\mathbf{i} + \sin\frac{s}{\sqrt{2}}\,\mathbf{j} + \frac{s}{\sqrt{2}}\,\mathbf{k} \quad s \ge 0$

TNB Frame



Unit Tangent Vector

Let C be a smooth curve represented by $\mathbf{r}$ on an interval I.

The unit tangent vector $\mathbf{T}$ is defined as $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

Curvature

直觀看法

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of $\mathbf{T}$.

定義

Let C be a smooth curve given by $\mathbf{r}(s)$, where s is the arc length parameter.

The curvature κ at s is $\kappa = \|\frac{d\mathbf{T}}{ds}\| = \|\mathbf{T}'(s)\|$ where $\mathbf{T}$ is the unit tangent vector.

- 計算方式：The curvature is easier to compute if it is expressed in terms of the parameter t instead of s so we use Chain Rule to write $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \Rightarrow \kappa = \|\frac{d\mathbf{T}}{ds}\| = \|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\|$.

  And $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, so we can also write $\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.

Theorem

If C is a smooth curve given by $\mathbf{r}(t)$,
then the curvature κ of C at t is $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.

Theorem

Suppose that C is twice differentiable. If C has nonzero tangent vector $\mathbf{r}'(t) = x'(t)\,\mathbf{i} + \mathbf{y}'(t)\,\mathbf{j}$, then $\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{([x'(t)]^2 + [y'(t)]^2)^{\frac{3}{2}}}$.

- For the graph of a twice differentiable function $y = f(x)$, $\kappa = \frac{|y''(x)|}{(1 + [y'(x)]^2)^{\frac{3}{2}}}$.

Principal Unit Normal

概念

Among the vectors orthogonal to the unit tangent vector $\mathbf{T}$, there is one of particular significance because it points the direction in which the curve is turning. Since $\mathbf{T}$ has constant length, the derivative $\frac{d\mathbf{T}}{ds}$ is orthogonal to $\mathbf{T}$. Therefore, if we divide $\frac{d\mathbf{T}}{ds}$ by its length, we obtain a unit vector $\mathbf{N}$ orthogonal to $\mathbf{T}$.

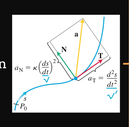
定義

At a point where $\|\mathbf{T}'(s)\| \neq \mathbf{0}$, the principal unit normal vector for a smooth curve in the plane is $\mathbf{N} = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$.

- 其它計算公式：If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, we can use the Chain Rule to calculate N directly:

  $\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{\frac{d\mathbf{T}}{dt}\frac{dt}{ds}}{|\frac{d\mathbf{T}}{dt}||\frac{dt}{ds}|} = \frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{dt}.$

Tangential and Normal Components of Acceleration



概念

When we study the motion of a partical, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal.
If we write $v = |\mathbf{v}|$ for the speed of the partical,

then $\mathbf{T}(t) = \frac{\frac{d\mathbf{r}}{dt}}{\|\frac{d\mathbf{r}}{dt}\|} = \frac{\mathbf{v}(t)}{v(t)}$ and so $\mathbf{v} = v\mathbf{T}$.

Differentiation with respect to t gives $\mathbf{a} = v'\mathbf{T} + v\mathbf{T}'$.

$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \Rightarrow \mathbf{T}' = \|\mathbf{T}'\|\mathbf{N} = \kappa v\mathbf{N} \quad (\|\mathbf{T}'\| = \kappa\|\mathbf{r}'(t)\| = \kappa v)$

Finally, we have $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$.

從這個式子可以知道，加速度一定在 $\mathbf{T}$ 和 $\mathbf{N}$ 所構成的平面上！

公式

If the acceleration vector is written as $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$,

then $\begin{cases} a_T = \frac{d^2s}{dt^2} = \frac{dv}{dt} = v' \\ a_N = \kappa(\frac{ds}{dt})^2 = \kappa v^2 \end{cases}$ are the tangential and normal scalar components of acceleration.

- $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a} = (a_T)^2 + (a_N)^2 \Rightarrow a_N = \sqrt{\|\mathbf{a}\| - (a_T)^2}$

  With this formula, we can find a_N without having to calculate κ first.

Binormal Vector

概念

Binormal vector is the tendency of your motion to "twist" out the plane created by $\mathbf{T}$ and $\mathbf{N}$ in the direction perpendicular to this plane. 用中文說就是，
$\mathbf{B}$ 是對於從 $\mathbf{T}$ 和 $\mathbf{N}$ 創造的平面上「扭曲」到與該平面垂直的方向的運動傾向！

定義

$\mathbf{B} = \mathbf{T} \times \mathbf{N}$

- The vector $\mathbf{B}$ is defined so that the ordered triple $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed system. And $\mathbf{B}$ is a unit vector since $\|\mathbf{B}\| = \|\mathbf{T}\|\,\|\mathbf{N}\| \sin\frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1.$

Torsion

概念

A space curve can also lift or "twist" out of the osculating plane at P.
Since $\mathbf{B}$ is normal to the osculating plane, $\frac{d\mathbf{B}}{ds}$ gives us information about how the osculating plane changes as P moves along C.
Since $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{B}$ (the latter has constant length) and $\mathbf{T}$,
It follows that $\frac{d\mathbf{B}}{ds}$ is orthogonal to the plane $\mathbf{B}$ and $\mathbf{T}$. In other words,
$\frac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, so $\frac{d\mathbf{B}}{ds}$ is a scalar multiple of $\mathbf{N}$. In symbols,
$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. (The negative sign in this equation is traditional.)
τ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve.

- 補充：Prove that $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$.

  $\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T} \times \mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \quad$ ($\mathbf{N}$ is the direction of $\frac{d\mathbf{T}}{ds}$)
  $= \mathbf{T} \times \frac{d\mathbf{N}}{ds}$

定義

$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau$

$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$

- Torsion is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write $\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt}$ so $\frac{d\mathbf{B}}{ds} = \frac{\frac{d\mathbf{B}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{B}'(t)}{\|\mathbf{r}'(t)\|}$.

  Finallly, we have $\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|}$.

Theorem

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\|\mathbf{v} \times \mathbf{a}\|^2}$$

Frenet-Serret Formulas

$\mathbf{T}'(s) = \kappa\,\mathbf{N}$
$\mathbf{N}'(s) = -\kappa\,\mathbf{T} + \tau\,\mathbf{B}$
$\mathbf{B}'(s) = -\tau\,\mathbf{N}$

想看證明可以去看 Vector Calculus by Susan Jane Colley。

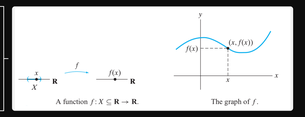
多變數純量函數極限與連續

極限

Visualizing Scalar Functions (Using Graph)

Single Variable Scalar Function

A function $f\colon X \subseteq \mathbb{R} \to \mathbb{R}$ takes a real number and returns another real number.
The graph of f is something that "lives" in $\mathbb{R}^2$. It consists of points (x, y) such that $y = f(x)$.
That is, Graph $f = \{(x, f(x)) \mid x \in X\} = \{(x, y) \mid x \in X, y = f(x)\}$.

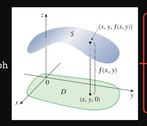


A function $f\colon X \subseteq \mathbf{R} \to \mathbf{R}$. The graph of f.

Two Variable Scalar Function

A function $f\colon X \subseteq \mathbb{R}^2 \to \mathbb{R}$ takes an ordered pair of real number and returns another real number.
The graph of f is something that "lives" in $\mathbb{R}^3$. It consists of points (x, y, z) such that $z = f(x, y)$.
That is, Graph $f = \{(x, y, f(x, y)) \mid x, y \in X\} = \{(x, y, z) \mid x, y \in X, z = f(x, y)\}$.

Graph



Level Curve

The level curve at height c of f is the curve in $\mathbb{R}^2$ defined by the equation $f(x, y) = c$, where c is a constant.
Symbolized, level curve at height $c = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$.

在畫等高線圖時，k 之間的差距通常取相同。
當 k 之間的差距相同時，Lever Curves 愈密，表示函數值變化愈大，也就是原本的曲面那部份會愈陡峭。

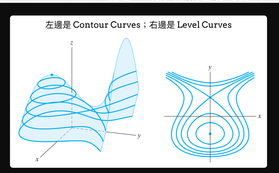
Contour Curve

The contour curve at height c of f is the curve in $\mathbb{R}^3$ defined by the two equations $z = f(x, y)$ and $z = c$. Symbolized, contour curve at height $c = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) = c\}$.

Section

The section of the graph of f by the plane
$\begin{cases} x = c \text{ (where } c \text{ is a constant) is the set of points } (x, y, z), \text{ where } z = f(x, y) \text{ and } x = c. \\ y = c \text{ (where } c \text{ is a constant) is the set of points } (x, y, z), \text{ where } z = f(x, y) \text{ and } y = c. \end{cases}$

左邊是 Contour Curves；右邊是 Level Curves



Functions of More Variables

If $f\colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ is a scalar function of n variable, then the **graph** of f is the subset of $\mathbb{R}^{n+1}$ given by
Graph $f = \{(\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in X\} = \{(x_1, \ldots, x_n, x_{n+1}) \mid (x_1, \ldots, x_n) \in X, x_{n+1} = f(x_1, \ldots, x_n)\}$.

The level set at height c of such a function is defined by Level set at height $c = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = c\}$.
While the graph of f is a subset of $\mathbb{R}^{n+1}$, a level set of f is a subset of $\mathbb{R}^n$. This makes it possible to
get some geometric insight into graphs of functions of three variables, even though we can't actually visualize them.

補充

The graph of a function of two variables does describe a surface in the "level set" sense. If a surface happens to be given by an equation of the form $z = f(x, y)$ for some appropriate function $f\colon X \subseteq \mathbb{R}^2 \to \mathbb{R}$, then we can move z to the opposite side, obtaining $f(x, y) - z = 0$. If we define a new function F of three variables by $F(x, y, z) = f(x, y) - z$, then the graph of f is precisely the level set at height 0 of F.

Conclusion: The graph of a function of two variables is a surface in $\mathbb{R}^3$ and is a "level set" of a function of three variables. However, not all level sets of functions of three variables are graphs of functions of two variables.

Definition

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Then to say $\lim\limits_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that given any $\varepsilon > 0$, you can find a $\delta > 0$ such that if $\mathbf{x} \in X$ and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $\|f(\mathbf{x}) - L\| < \varepsilon$.

若極限值存在則唯一

為了幫助你理解，以下是其他微積分用書常見的雙變數函數極限定義。
Let f be a function defined on an open disk（圓盤）centered at (a, b) [except possibly at (a, b)].

Limit of Scalar Function of Two Variable

因為其他教科書常用雙變數純量函數舉例，故以此為例。

Definition

We say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$

if for all $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

證明不存在

If $\lim\limits_{(x,y)\to(a,b) \text{ along path } C_1} f(x, y) \neq \lim\limits_{(x,y)\to(a,b) \text{ along path } C_2} f(x, y)$,
then $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exits.

範例

Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ exists?

For the path $x = 0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4} = 0$

For the path $y = mx$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4} = \lim\limits_{(x,y)\to(0,0)} \dfrac{m^2 x}{1 + m^4 x^2} = 0$

Though f has the same limiting value along every line through the origin.
But that doesn't show the given limit is 0, for if we now consider the following path.

For the path $x = y^2$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4} = \lim\limits_{(x,y)\to(0,0)} \dfrac{y^4}{2y^4} = \dfrac{1}{2}$

$\therefore$ The limit doesn't exist.

Proposition

Limit Laws

Let $f, g\colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ be scalar-valued functions, and let $k \in \mathbb{R}$ be a scalar.

$\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L \quad \lim\limits_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$

- $\lim\limits_{\mathbf{x}\to\mathbf{a}} (f \pm g)(\mathbf{x}) = L \pm M$
- $\lim\limits_{\mathbf{x}\to\mathbf{a}} kf(\mathbf{x}) = kL$
- $\lim\limits_{\mathbf{x}\to\mathbf{a}} (fg)(\mathbf{x}) = LM$
- $\lim\limits_{\mathbf{x}\to\mathbf{a}} [f(\mathbf{x})]^n = L^n$
- $\lim\limits_{\mathbf{x}\to\mathbf{a}} (\dfrac{f}{g})(\mathbf{x}) = \dfrac{L}{M} \quad (M \neq 0)$
- $\lim\limits_{\mathbf{x}\to\mathbf{a}} \sqrt[n]{f(\mathbf{x})} = \sqrt[n]{L}$ where $n \in \mathbb{R}$
  If n is even, we assume that $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \geq 0$.

Theorem

Squeeze Theorem

Let $A \subseteq \mathbb{R}^n$ be a set, and $f, g\colon A \to \mathbb{R}^m$ be scalar functions.
Suppose that $x_0 \in A$, $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in A - \{\mathbf{x}_0\}$, and $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ exists.
Then $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$ exists and $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$.

If $f(x, y) \leq g(x, y) \leq h(x, y)$ when (x, y) is near (a, b) [except possibly at (a,b)]
and $\lim\limits_{(x,y)\to(a,b)} f(x, y) = \lim\limits_{(x,y)\to(a,b)} h(x) = L \Rightarrow \lim\limits_{(x,y)\to(a,b)} g(x) = L$

範例

求 $\lim\limits_{(x,y)\to(0,0)} x\sin(x + y)$

$\because 0 \leq |x\sin(x + y)| \leq |x|$ and $\lim\limits_{(x,y)\to(0,0)} |x| = 0$

$\therefore \lim\limits_{(x,y)\to(0,0)} |x\sin(x + y)| = 0 \Rightarrow \lim\limits_{(x,y)\to(0,0)} x\sin(x + y) = 0$

連續

Definition

Let $f\colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ and let $\mathbf{a} \in X$. Then f is said to be continuous at $\mathbf{a}$ if $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.
If f is continuous at all points of its domain X, then we simply say that f is continuous.

Proposition

If $f, g\colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ that are continuous at $\mathbf{a} \in X$ and c is a constant, then the following functions are also continuous at $\mathbf{a}$:

1. $f \pm g$ 2. cf 3. fg 4. $\dfrac{f}{g}$ if $g(a, b) \neq 0$

If g is continuous at the point $\mathbf{x_0}$ and f is continuous at the number $g(\mathbf{x}_0)$,
then the composition $f \circ g$ is continuous at the point $\mathbf{x}_0$.

多變數純量函數偏導數（概念篇）

Partial Derivative 偏導數

概念 偏導數是一種只針對多變數函數某一個變數微分而得的導數，且微分的過程裡將其他的變數都視為常數。
我們從前學習單變數函數時沒有偏導數的概念，但在多變數函數的世界裡，偏導數是個很重要的工具。
為了簡化我們的學習，我們先從雙變數函數 $z=f(x,y)$ 開始考慮，只要同學們充分了解 $z=f(x,y)$ 的偏導數如何運算，推廣到 $y=f(x_1,x_2,\cdots,x_n)$ 做法也會相同！

Definition

If u is a function of n variables, $u=f(x_1,x_2\cdots,x_n)$, its partial derivative with respect to the ith variable x_i is $\dfrac{\partial u}{\partial x_i}=\lim\limits_{h\to 0}\dfrac{f(x_1,\ldots,x_i+h,\ldots,x_n)-f(x_1,\ldots,x_n)}{h}$ and we also write $\dfrac{\partial u}{\partial x_i}=\dfrac{\partial f}{\partial x_i}=f_{x_i}=D_{x_i}f$.

範例：雙變數函數偏導數

If f is a function of two variables, its partial derivatives are the function f_x and f_y defined by

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

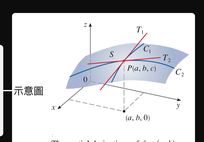
$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+k)-f(x,y)}{h}$$

幾何意義

Let $\begin{cases}(1)\text{ The surface } S \text{ be the graph of } f(x,y)\\(2)\text{ The curve } C_1 \text{ be intersection of the plane } y=b \text{ and } S\\(3)\text{ The curve } C_2 \text{ be intersection of the plane } x=a \text{ and } S\end{cases}$

Suppose that $f(a,b)=c$ then $\begin{cases}f_x(a,b)\text{ is the slope (斜率) of the tangent line at } (a,b,c) \text{ to } C_1\,.\\f_y(a,b)\text{ is the slope of the tangent line at } (a,b,c) \text{ to } C_2\,.\end{cases}$

示意圖



The partial derivatives of f at (a,b) are the slopes of the tangents to C_1 and C_2.

Partial Derivative & Continuity

雖然單變數函數導數存在保證連續，
但是多變數函數偏導數存在無法保證連續！

例子（以雙變數函數為例）：
在原點偏導數存在，
但極限不存在，
所以一定不連續。

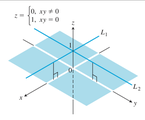


FIGURE 14.21 The graph of $f(x,y)=\begin{cases}0, & xy\neq 0\\1, & xy=0\end{cases}$

Higher Derivative（以二階偏導數為例）

Let $z=f(x,y)$.

Differentiate twice w.r.t x: $\dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial x})=\dfrac{\partial^2 f}{\partial x^2}=f_{xx}=(f_x)_x=f_{11}$

Differentiate twice w.r.t y: $\dfrac{\partial}{\partial y}(\dfrac{\partial f}{\partial y})=\dfrac{\partial^2 f}{\partial y^2}=f_{yy}=(f_y)_y=f_{22}$

Differentiate first w.r.t x and then w.r.t y: $\dfrac{\partial}{\partial y}(\dfrac{\partial f}{\partial x})=\dfrac{\partial^2 f}{\partial y\partial x}=f_{xy}=(f_x)_y=f_{12}$

Differentiate first w.r.t y and then w.r.t x: $\dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial y})=\dfrac{\partial^2 f}{\partial x\partial y}=f_{yx}=(f_y)_x=f_{21}$

Proposition — Partial Differentiation Rules

$$\begin{cases}(cf)_{x_i}=cf_{x_i} \quad c\in\mathbb{R}\\(f+g)_{x_i}=f_{x_i}+g_{x_i}\\(f\cdot g)_{x_i}=f_{x_i}g+g_{x_i}f\\(\dfrac{f}{g})_{x_i}=\dfrac{f_{x_i}g-g_{x_i}f}{g^2}\end{cases}$$

Theorem — Clairaut's Theorem

Let $z=f(x,y)$. If f_{xy} and f_{yx} are continuous on an open disk D then $f_{xy}(x,y)=f_{yx}(x,y) \quad \forall\,(x,y)\in D$.

多變數函數可微

Definition

Denote $\begin{cases}\text{the increment of } x \text{ by } \Delta x\\\text{the increment of } y \text{ by } \Delta y\text{ , so } \Delta z=f(a+\Delta x,b+\Delta y)-f(a,b).\\\text{the increment of } z \text{ by } \Delta z\end{cases}$

Then, $f(x,y)$ is differentiable at (a,b) if $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$, where $\varepsilon_1,\varepsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$.

我們通常不會用定義去看一個點可不可微，我們會用下面的定理去看！（以雙變數函數為例）

Let $z=f(x,y)$ be a function of two variables.
If the partial derivatives f_x and f_y exists near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Tangent Plane

If $z=f(x,y)$ is differentiable at (x_0,y_0), then there exists a tangent plane at the point (x_0,y_0,z_0), and the equation of the tangent plane is $z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$.

補充：Scalar parametric equations for the line normal to the tangent plane at the point (x_0,y_0,z_0) is $\begin{cases}x=x_0+f_x(x_0,y_0)\,t\\y=y_0+f_y(x_0,y_0)\,t\\z=z_0+(-1)\,t\end{cases}$.

Proposition

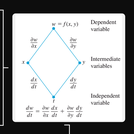
可微必連續 If f is differentiable at $\mathbf{x}$, then f is continuous at $\mathbf{x}$.

Chain Rule

簡寫符號說明：
(2,1) 表示兩個 Intermediate Variables 和一個 Independent Variable

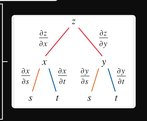
- **Case 1: (2,1)**

Suppose $w=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable function of t.
Then w is a differentiable function of t, and $\dfrac{dw}{dt}=\dfrac{\partial w}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$.



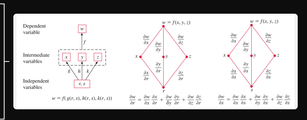
- **Case 2: (2,2)**

Suppose $z=f(x,y)$ is a differentiable function of x and y, where $x=g(s,t)$ and $y=h(s,t)$ are both differentiable function of s and t.
Then, $\dfrac{\partial z}{\partial s}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ $\quad\dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$.



- **Case 3: (3,2)**

Suppose $w=f(x,y,z)$ is a differentiable function of x and y and z, where $x=g(r,s)$ and $y=h(r,s)$ and $z=j(r,s)$ are both differentiable function of r and s.
Then $\dfrac{\partial w}{\partial r}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial w}{\partial s}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial s}$



- **General Version**

Suppose that u is a differentiable function of the n variables $x_1,x_2,\ldots,x_n$ and each x_j is a differentiable function of the m variables $t_1,t_2,\ldots,t_m$ and $\dfrac{\partial u}{\partial t_i}=\dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial t_i}+\dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial t_i}+\cdots+\dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}$ for each $i=1,2,\ldots,m$.

- **Implicit Differentiation**

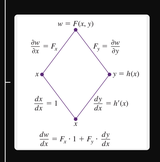
之前有提過，但那時無法解釋，現在用雙變數函數當例子，回憶一下！
Suppose that $w=F(x,y)$ is a continuously differentiable function. If y is a differentiable function of x that satisfies $w=0$, then we can find the derivative of y respect to x without first having to express y explicitly in terms of x.
The process by which we do this is called implicit differentiation.

雙變數函數

According to the Chain Rule for functions of one independent variable and two intermediate variables, suppose that $\begin{cases}\text{The functions } F(x,y) \text{ is differentiable.}\\\text{The equation } F(x,y)=0 \text{ defines } y \text{ implicitly as a differentiable function of } x.\end{cases}$

Since $w=F(x,y)=0$, the derivative $\dfrac{dw}{dx}$ must be 0. Computing the derivative from the chain rule, we find $0=\dfrac{dw}{dx}=F_x\dfrac{dx}{dx}+F_y\dfrac{dy}{dx}=F_x\cdot 1+F_y\dfrac{dy}{dx}$.

If $F_y=\dfrac{\partial w}{\partial y}\neq 0$, we can solve this to get $\dfrac{dy}{dx}=-\dfrac{F_x(x,y)}{F_y(x,y)}$.



三變數函數 If the equation $F(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x(x,y,z)}{F_z(x,y,z)}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y(x,y,z)}{F_z(x,y,z)}$, $\quad F_z(x,y,z)\neq 0$.

Directional Derivative

Definition

For each unit vector (單位向量) $\mathbf{u}$, the directional derivative (方向導數) of f at $\mathbf{x}$ in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{x})=\lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{u})-f(\mathbf{x})}{h}\text{ if the limit exists.}$$

⚠ 偏導數可看成是方向導數的特例。

Directional Derivative & Continuity

Let f be the scalar function defined on $\mathbb{R}^2$ as follows: $f(x,y)=\dfrac{xy^2}{x^2+y^4}$ if $x\neq 0$, $f(0,y)=0$.
Let $\mathbf{c}=(0,0)$ and let $\mathbf{y}=(a,b)$ be any vector. If $a\neq 0$ and $h\neq 0$, we have $\lim\limits_{h\to 0}\dfrac{f(\mathbf{c}+h\mathbf{y})-f(\mathbf{c})}{h}=\lim\limits_{h\to 0}\dfrac{ab^2}{a^2+h^2b^4}=\dfrac{b^2}{a}$. Therefore, the direcrtional derivative exists for all direction and equals 0. However, at each point of the parabola $x=y^2$ (except the origin) the function has the value $\dfrac{1}{2}$. Thus, the function is not continuous at $\mathbf{c}$.

雖然單變數函數導數存在保證連續，
但是多變數函數方向導數存在無法保證連續！

Theorem If f is a differentiable at $\mathbf{x}$, then $\begin{cases}f\text{ has a directional derivative at } \mathbf{x} \text{ in every direction.}\\\text{for each unit vector } \mathbf{u}, \mathbf{D_u}f(\mathbf{x})=\langle f_x(x,y),f_y(x,y)\rangle\cdot\mathbf{u}.\end{cases}$

Gradient

Definition

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by $\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle$

Differentiation Rules

Suppose that α is a constant and f and g are both differentiable.

$$\begin{cases}\nabla[f(\mathbf{x})+g(\mathbf{x})]=\nabla f(\mathbf{x})+\nabla g(\mathbf{x})\\\nabla[\alpha f(\mathbf{x})]=\alpha\nabla f(\mathbf{x})\\\nabla[f(\mathbf{x})g(\mathbf{x})]=f(\mathbf{x})\nabla g(\mathbf{x})+g(\mathbf{x})\nabla f(\mathbf{x})\end{cases}$$

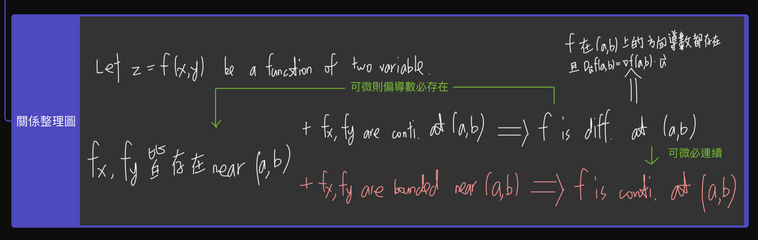
Theorem

- Mean-Value Theorem: If f is differentiable at each point of line segment $\overline{\mathbf{ab}}$, then there exists on that line segment a point $\mathbf{c}$ between $\mathbf{a}$ and $\mathbf{b}$ such that $f(\mathbf{b})-f(\mathbf{a})=\nabla f(\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})$.
- Let U be an open connected set and let f be a differentiable function on U. If $\nabla f(\mathbf{x})=\mathbf{0}$ for all $\mathbf{x}$ in U, then f is constant on U.
  - Corollary: Let U be an open connected set and let f and g be functions differentiable on U. If $\nabla f(\mathbf{x})=\nabla g(\mathbf{x})$, then f and g differ by a constant on U.

補充：對位置函數取梯度公式

Let $r(x,y,z)=\sqrt{x^2+y^2+z^2}$; $\mathbf{r}=x\,\mathbf{i}+y\mathbf{j}+z\,\mathbf{k}$.
$\nabla r^n=nr^{n-2}\mathbf{r} \quad \forall\, n\in\mathbb{Z},\mathbf{r}\neq 0$
If n is positive and even, the results also holds at $\mathbf{r}=0$.

關係整理圖



多變數純量函數偏導數（應用篇）

Application of Gradient

算近似值 — Linear Approximation

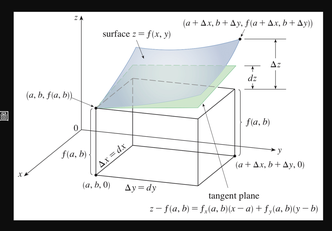
之前有提到可微則存在切平面，我們可以用切平面作線性近似，說明如下。

If f is differentiable at (a,b),
then the approximating function $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is the linearization of f at (a,b).
The approximation $f(x,y) \approx L(x,y)$ is called the linear approximation of f at (a,b).

Total Differential

For a differentiable function of two variables, $z = f(x,y)$,
we define the differentials dx and dy to be independent variables; that is, they can be given any values.
Then the differential dz, also called the total differential, is defined by $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$.

示意圖



Relationship with Directional Derivative

Let f be a differentiable function. $\mathbf{D_u} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.

If $\nabla f(\mathbf{x}) \neq \mathbf{0}$: $\begin{cases} \text{1. } \nabla f(\mathbf{x}) \text{ points in the direction of maximum rate of change of } f \text{ at } \mathbf{x}, \\ \quad \text{while -}\nabla f(\mathbf{x}) \text{ points in the direction of minimum rate of change of } f \text{ at } \mathbf{x}. \\ \text{2. The maximum rate of change is } |\nabla f(\mathbf{x})|\text{, and the minimum rate of change is -}|\nabla f(\mathbf{x})|. \end{cases}$

If $\nabla f(\mathbf{x}) = \mathbf{0}$: $D_u f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{u}$.

這些都是要在可微的情況下才會發生！

Level Curve and Level Surface（Salas 的書都有解釋，可以去看！）

梯度垂直等高線和等高面

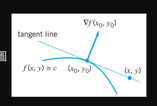
At each point of the domain, the gradient vector ∇f, if not $\mathbf{0}$,
is prependicular to the level curve or level surface of f through that point.
⚠ Salas 的書有證明，可以去翻翻看！

Level Curve $f(x,y) = c$

Tangent Line

Equation $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$

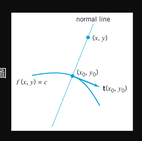
示意圖



Normal Line

Equation $f_y(x_0,y_0)(x-x_0) - f_x(x_0,y_0)(y-y_0) = 0$

示意圖

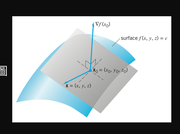


Level Surface $f(x,y,z) = c$

Tangent Plane

Equation $f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$

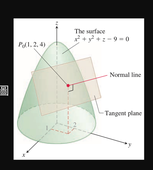
示意圖



Normal Line

Equation
$$\begin{aligned} x &= x_0 + f_x(x_0,y_0,z_0)\,t \\ y &= y_0 + f_y(x_0,y_0,z_0)\,t \\ z &= z_0 + f_z(x_0,y_0,z_0)\,t \end{aligned}$$

示意圖



補充：找雙變數函數 Graph 的 Tangent Plane 和 Normal Line 的方法

A surface of the form $z = g(x,y)$ can be written in the form $f(x,y,z) = 0$ by setting
$f(x,y,z) = g(x,y) - z$. If g is differentiable, so is f. Moreover, $\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = \frac{\partial g}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y,z) = \frac{\partial g}{\partial y}(x,y) \\ \frac{\partial f}{\partial z}(x,y,z) = -1 \end{cases}$.

Tangent Plane

The tangent plane at (x_0,y_0,z_0) has equation $g_x(x_0,y_0)\,(x-x_0) + g_y(x_0,y_0)\,(y-y_0) + (-1)(z-z_0) = 0$,
which we can rewrite as $z - z_0 = g_x(x_0,y_0)\,(x-x_0) + g_y(x_0,y_0)\,(y-y_0)$. If $\nabla g(x_0,y_0) = \mathbf{0}$,
then both partials of g are zero at (x_0,y_0) and the equation reduces to $z = z_0$. In this case,
the tangent plane is horizontal.

Normal Line

$$\begin{aligned} x &= x_0 + g_x(x_0,y_0)\,t \\ y &= y_0 + g_y(x_0,y_0)\,t \\ z &= z_0 + (-1)\,t \end{aligned}$$

極值介紹

Absolute Extrema 絕對極值

Suppose that f is a function of several variables.
The function f is said to have a $\begin{cases} \text{absolute maximum} \\ \text{absolute minimum} \end{cases}$ at $\mathbf{x}_0$ provided that $\begin{cases} f(\mathbf{x}_0) \geq f(\mathbf{x}) \\ f(\mathbf{x}_0) \leq f(\mathbf{x}) \end{cases}$
for all $\mathbf{x}$ in the domain of f. The absolute maxima and minima of f comprise the absolute extreme values of f.

Local Extrema 局部極值

Suppose that f is a function of several variables and $\mathbf{x}_0$ is an interior point of the domain.
The function f is said to have a $\begin{cases} \text{local maximum} \\ \text{local minimum} \end{cases}$ at $\mathbf{x}_0$ provided that $\begin{cases} f(\mathbf{x}_0) \geq f(\mathbf{x}) \\ f(\mathbf{x}_0) \leq f(\mathbf{x}) \end{cases}$
for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$. The local maxima and minima of f comprise the local extreme values of f.

找極值

找相對極值

Theorem

If f has a local maximum or minimum at $\mathbf{x}_0$,
then $\mathbf{x}_0$ is a critical point of f.

Critical Points

Let f defined on an open region R containing (x_0,y_0).
The point $\mathbf{x}_0$ is a critical point of f if one of the following is true.
1. $\nabla f(\mathbf{x}_0) = \mathbf{0}$.
2. $\nabla f(\mathbf{x}_0)$ does not exist.
我們稱 Critical Points 中，$\nabla f(\mathbf{x}_0) = \mathbf{0}$ 的那些 $\mathbf{x}_0$ 為 Stationary Points。
如果一個 Stationary Points 沒有極值，就稱它為 Saddle Point（馬鞍點）。

找相對極值方法 Second Derivatives Test

Let f have continuous second partial derivatives on an open region containing a point (a,b) and $\nabla f(a,b) = \langle 0,0 \rangle$.
Consider the matrix $\begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$ and the quantity $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(1) If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b) .
(2) If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b) .
(3) If $D < 0$, then $(a,b,f(a,b))$ is a saddle point.
(4) If $D = 0$, then the test fails.
⚠ 想知道更多，可以去看陽交大蘇承芳老師的講義！

跟單變數的時候一樣，為了知道 Critical Points 有沒有相對極值，
我們需要 Second Derivatives Test 來告訴我們，
畢竟不是所有的 Critical Points 都有相對極值！

找絕對極值

Extreme Value Theorem

If f is continuous on a closed and bounded set D, then on that set
f takes on an absolute maximum and an absolute minimum.

Bounded

A subset of the plane or three-space is said to be bounded
provided there exists a positive number R such that $\|\mathbf{x}\| \leq R$ for all $\mathbf{x} \in S$.

找絕對極值方法

To find the absolute extrema of a continuous function f on a closed, bounded set D:
1. Find the value of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. Compare the values of 1 and 2.（這些值中最大的就是絕對極大值；最小的就是絕對極小值！）

拉格朗日乘數法（適用於有條件的情況下）

在一個限制下

定理

Let f and g have continuous first partial derivatives such that
f has an extremum at a point (x_0,y_0,z_0) on the smooth constraint curve $g(x,y,z) = c$.
If $\nabla g(x_0,y_0,z_0) \neq \langle 0,0,0 \rangle$, then there is a real number λ such that $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$.

計算方法

(1) Find all value of x, y, z, and λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = c$.
(2)將上面所得到的所有的 x, y, z 全部代入 $f(x,y,z)$ 裡把值算出來。這些值中最大的就是 f 的最大值，最小的就是 f 的最小值。

在兩個限制下

定理

If $f(x,y,z)$ subject to the side constraints $g(x,y,z) = k$ and $h(x,y,z) = c$ and f has an extremum at a point (x_0,y_0,z_0),
then $\nabla f(x_0,y_0,z_0)$ is in the plane determined by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$.
If $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$ are not parallel and not $\langle 0,0,0 \rangle$, then there exist λ and μ such that $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$.

計算方法

(1) Find all value of x, y, z, λ, and μ such that $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = c \\ h(x,y,z) = k \end{cases}$

(2)將上面所得到的所有的 x, y, z 全部代入 $f(x,y,z)$ 裡把值算出來。這些值中最大的就是 f 的最大值，最小的就是 f 的最小值。

多變數純量函數重積分

Double Integral

Multiple-Sigma

Proposition

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha\, a_{ij} = \alpha\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij} \pm b_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} \pm \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}$$

Sums of Powers

$$\sum_{i=1}^{n} c = cn \quad \text{if } c \text{ is a constant}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Riemann Integral

Let $R = [a,b]\times[c,d]$ be a rectangle in the plane, and $f\colon R \to \mathbb{R}$ be a function.
f is said to be Riemann integrable on R if there exists a real number V such that
$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that if $\mathcal{P}$ is partition of R satisfying $\|\mathcal{P}\| < \delta$, then any
Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(V-\varepsilon, V+\varepsilon)$.
Such a number V is called Riemann integral or double integral of f on R
and is denoted by $\displaystyle\iint_R f(x,y)\,dA$.

Riemann Integral 存在則唯一！

連續必可積

Let R be closed and bounded region in the plane, and $f\colon R \to \mathbb{R}$ be a function.
If R has area and f is continuous on R, then f is Riemann integrable on R.

Iterated Integral

在多變數中，我們用 Iterated Integrals 來算重積分，方法如下：
將某些變數視為給定的常數，只針對特定的一個變數做積分。我們假設 $f(x,y)$ 為一個定義在 $\mathbb{R}^2$ 上的雙變數函數，
我們可以先將 y 視為常數，這樣就可以對變數 x 在區間 I 上做積分 $\displaystyle\int_I f(x,y)\,dx$，這樣積分完後會是 y 的函數 $F(y)$。
接著，我們對變數 y 在區間 J 上對 $F(y)$ 再次積分，也就是計算 $\displaystyle\int_J F(y)\,dy$，就可以得到以下的積分形式：$\displaystyle\int_J F(y)\,dy = \int_J\left\{\int_I f(x,y)\,dx\right\}dy = \int_J\int_I f(x,y)\,dxdy$.
綜上所述，形如 $\displaystyle\int_J\int_I f(x,y)\,dxdy$ 的積分形式就叫做 Iterated Integrals，它的積分方式是一層一層積，由內往外積，也可稱為疊積分。

Fubini's Theorem（用來交換積分順序）

Let $R = [a,b]\times[c,d]$. If f satisfies $\begin{cases} f \text{ is continuous on } R \\ f \text{ is bounded on } R \text{ and is discontinuous only at a finite number of smooth curves} \end{cases}$

then $\displaystyle\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy.$

If f(x,y) can be written in $g(x)h(y)$,
then $\displaystyle\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx\int_c^d h(y)\,dy$
where $R = [a,b]\times[c,d]$

Average Value

$$\frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA$$

Double Integrals over General Region

除了積分矩形區域，我們也想積分非矩形區域！
目前我們只能處理兩種區域。

Type 1

$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$
If f is continuous on D,
then $\displaystyle\iint_D f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$

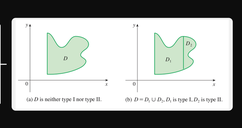
Type 2

$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$
If f is continuous on D,
then $\displaystyle\iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$

Properties

如果不是以上兩種區域，可以考慮進行切割，並套用以下公式。

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Substitution in Multiple Integrals

在單變數函數積分的時候，有介紹過變數變換的作法，
現在要來介紹在重積分時，變數變換的作法！

Jacobian

The jacobian of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Jacobian 在幾何上可以看成是，變數變換後的區域微小面積（或體積）與原來區域微小面積（或體積）之間的比例因子。

可以看看這個影片！ → Change of Variables & The Jacobian by Dr. Trefor Bazett

Theorem

Suppose that $\begin{cases} T \text{ is a } C^1 \text{ transformation (即 } \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v} \text{ are continuous).} \\ \text{Jacobian of } T \text{ doesn't equal 0.} \\ T \text{ is one-to-one (即反函數 } T^{-1} \text{ 存在), except perhaps on the boundary of S.} \end{cases}$ holds,

f is continuous on R, and T maps region S in the $uv-$plane onto the region R in $xy-$plane.

Then $\displaystyle\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_S f(g(u,v),h(u,v))\left|\det(J)\right| du\,dv$.

Double Integral in Polar Coordinates

坐標變換是一種變數變換。

可以看看這個影片！

Change of Variables and the Jacobian by Serpentine Integral

For $\alpha, \beta \in \mathbb{R}$ satisfy $0 \le \beta - \alpha \le 2\pi$:
If f is continuous on polar rectangle $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ 0 \le a \le r \le b\}$,
then $\displaystyle\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

For $\alpha, \beta \in \mathbb{R}$ satisfy $0 \le \beta - \alpha \le 2\pi$:
If f is continuous on polar rectangle $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ 0 \le g_1(\theta) \le r \le g_2(\theta)\}$,
then $\displaystyle\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Surface Area

The area of the surface with equation $z = f(x,y)$, $(x,y) \in D$, where f_x and f_y are continuous is

$$A(S) = \iint_D \sqrt{\|\nabla f(x,y)\|^2 + 1}\,dA = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$$

Triple Integral

Definition

Let $Q = [a,b]\times[c,d]\times[r,s]$ be a cube in the space, and $f\colon Q \to \mathbb{R}$ be a function.
f is said to be Riemann integrable on Q if there exists a real number I such that
$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that if $\mathcal{P}$ is partition of Q satisfying $\|\mathcal{P}\| < \delta$, then any
Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(I-\varepsilon, I+\varepsilon)$.
Such a number I is called Riemann integral or triple integral of f on Q
and is denoted by $\displaystyle\iiint_Q f(x,y,z)\,dV$.

Riemann Integral 存在則唯一！

Fubini's Theorem

Let $B = [a,b]\times[c,d]\times[r,s]$. If f satisfies $\begin{cases} f \text{ is continuous on } B \\ f \text{ is bounded on } B \text{ and is discontinuous only at a finite number of smooth surfaces'} \end{cases}$

$$\begin{aligned}\text{then } \iiint_B f(x,y,z)\,dV &= \int_a^b\int_c^d\int_r^s f(x,y,z)\,dzdydx = \int_a^b\int_r^s\int_c^d f(x,y,z)\,dzdxdy \\ &= \int_c^d\int_a^b\int_r^s f(x,y,z)\,dydzdx = \int_c^d\int_r^s\int_a^b f(x,y,z)\,dydxdz \\ &= \int_r^s\int_c^d\int_a^b f(x,y,z)\,dxdydz = \int_r^s\int_a^b\int_c^d f(x,y,z)\,dxdzdy.\end{aligned}$$

Triple Integrals over General Region

Type 1

Let $E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdydx$.

Type 2

Let $E = \{(x,y,z) \mid c \le y \le d,\ g_1(y) \le x \le g_2(y),\ h_1(x,y) \le z \le h_2(x,y)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_c^d\int_{g_1(y)}^{g_2(y)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdxdy$.

Type 3

Let $E = \{(x,y,z) \mid c \le y \le d,\ g_1(y) \le z \le g_2(y),\ h_1(y,z) \le x \le h_2(y,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_c^d\int_{g_1(y)}^{g_2(y)}\int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dxdzdy$.

Type 4

Let $E = \{(x,y,z) \mid r \le z \le s,\ g_1(z) \le y \le g_2(z),\ h_1(y,z) \le x \le h_2(y,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_r^s\int_{g_1(z)}^{g_2(z)}\int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dxdydz$.

Type 5

Let $E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le z \le g_2(x),\ h_1(x,z) \le y \le h_2(x,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dydzdx$.

Type 6

Let $E = \{(x,y,z) \mid r \le z \le s,\ g_1(z) \le x \le g_2(z),\ h_1(x,z) \le y \le h_2(x,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_r^s\int_{g_1(z)}^{g_2(z)}\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dydxdz$.

Substitution in Multiple Integrals

在單變數函數積分的時候，有介紹過變數變換的作法，
現在要來介紹在重積分時，變數變換的作法！

Jacobian

The jacobian of the transformation T given by $x = g(u,v,w)$ and $y = h(u,v,w)$ and $z = k(u,v,w)$ is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Jacobian 在幾何上可以看成是，變數變換後的區域微小面積（或體積）與原來區域微小面積（或體積）之間的比例因子。

Theorem

Suppose that $\begin{cases} T \text{ is a } C^1 \text{ transformation (即 All partial derivatives are continuous).} \\ \text{Jacobian of } T \text{ doesn't equal 0.} \\ T \text{ is one-to-one (即反函數 } T^{-1} \text{ 存在), except perhaps on the boundary of S.} \end{cases}$ holds,

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Then $\displaystyle\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \iiint_S f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\det(J)\right| du\,dv\,dw$.

Cylindrical coordinates

坐標轉換

柱坐標→直角坐標 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

直角坐標→柱坐標 $\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$

計算公式

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Spherical coordinates

坐標轉換

球坐標→直角坐標 $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$

直角坐標→球坐標 $\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\theta = \frac{y}{x} \\ \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$

計算公式

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$