多變數純量函數重積分

Double Integral

Multiple-Sigma

Proposition

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha\, a_{ij} = \alpha\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij} \pm b_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} \pm \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}$$

Sums of Powers

$$\sum_{i=1}^{n} c = cn \quad \text{if } c \text{ is a constant}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Riemann Integral

Let $R = [a,b]\times[c,d]$ be a rectangle in the plane, and $f\colon R \to \mathbb{R}$ be a function.
f is said to be Riemann integrable on R if there exists a real number V such that
$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that if $\mathcal{P}$ is partition of R satisfying $\|\mathcal{P}\| < \delta$, then any
Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(V-\varepsilon, V+\varepsilon)$.
Such a number V is called Riemann integral or double integral of f on R
and is denoted by $\displaystyle\iint_R f(x,y)\,dA$.

Riemann Integral 存在則唯一！

連續必可積

Let R be closed and bounded region in the plane, and $f\colon R \to \mathbb{R}$ be a function.
If R has area and f is continuous on R, then f is Riemann integrable on R.

Iterated Integral

在多變數中，我們用 Iterated Integrals 來算重積分，方法如下：
將某些變數視為給定的常數，只針對特定的一個變數做積分。我們假設 $f(x,y)$ 為一個定義在 $\mathbb{R}^2$ 上的雙變數函數，
我們可以先將 y 視為常數，這樣就可以對變數 x 在區間 I 上做積分 $\displaystyle\int_I f(x,y)\,dx$，這樣積分完後會是 y 的函數 $F(y)$。
接著，我們對變數 y 在區間 J 上對 $F(y)$ 再次積分，也就是計算 $\displaystyle\int_J F(y)\,dy$，就可以得到以下的積分形式：$\displaystyle\int_J F(y)\,dy = \int_J\left\{\int_I f(x,y)\,dx\right\}dy = \int_J\int_I f(x,y)\,dxdy$.
綜上所述，形如 $\displaystyle\int_J\int_I f(x,y)\,dxdy$ 的積分形式就叫做 Iterated Integrals，它的積分方式是一層一層積，由內往外積，也可稱為疊積分。

Fubini's Theorem（用來交換積分順序）

Let $R = [a,b]\times[c,d]$. If f satisfies $\begin{cases} f \text{ is continuous on } R \\ f \text{ is bounded on } R \text{ and is discontinuous only at a finite number of smooth curves} \end{cases}$

then $\displaystyle\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy.$

If f(x,y) can be written in $g(x)h(y)$,
then $\displaystyle\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx\int_c^d h(y)\,dy$
where $R = [a,b]\times[c,d]$

Average Value

$$\frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA$$

Double Integrals over General Region

除了積分矩形區域，我們也想積分非矩形區域！
目前我們只能處理兩種區域。

Type 1

$D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$
If f is continuous on D,
then $\displaystyle\iint_D f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$

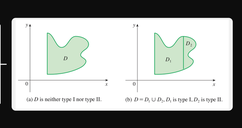
Type 2

$D = \{(x,y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$
If f is continuous on D,
then $\displaystyle\iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$

Properties

如果不是以上兩種區域，可以考慮進行切割，並套用以下公式。

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Substitution in Multiple Integrals

在單變數函數積分的時候，有介紹過變數變換的作法，
現在要來介紹在重積分時，變數變換的作法！

Jacobian

The jacobian of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Jacobian 在幾何上可以看成是，變數變換後的區域微小面積（或體積）與原來區域微小面積（或體積）之間的比例因子。

可以看看這個影片！ → Change of Variables & The Jacobian by Dr. Trefor Bazett

Theorem

Suppose that $\begin{cases} T \text{ is a } C^1 \text{ transformation (即 } \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v} \text{ are continuous).} \\ \text{Jacobian of } T \text{ doesn't equal 0.} \\ T \text{ is one-to-one (即反函數 } T^{-1} \text{ 存在), except perhaps on the boundary of S.} \end{cases}$ holds,

f is continuous on R, and T maps region S in the $uv-$plane onto the region R in $xy-$plane.

Then $\displaystyle\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_S f(g(u,v),h(u,v))\left|\det(J)\right| du\,dv$.

Double Integral in Polar Coordinates

坐標變換是一種變數變換。

可以看看這個影片！

Change of Variables and the Jacobian by Serpentine Integral

For $\alpha, \beta \in \mathbb{R}$ satisfy $0 \le \beta - \alpha \le 2\pi$:
If f is continuous on polar rectangle $R = \{(r,\theta)\,|\,\alpha \le \theta \le \beta,\ 0 \le a \le r \le b\}$,
then $\displaystyle\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

For $\alpha, \beta \in \mathbb{R}$ satisfy $0 \le \beta - \alpha \le 2\pi$:
If f is continuous on polar rectangle $R = \{(r,\theta)\,|\,\alpha \le \theta \le \beta,\ 0 \le g_1(\theta) \le r \le g_2(\theta)\}$,
then $\displaystyle\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Surface Area

The area of the surface with equation $z = f(x,y)$, $(x,y) \in D$, where f_x and f_y are continuous is

$$A(S) = \iint_D \sqrt{\|\nabla f(x,y)\|^2 + 1}\,dA = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$$

Triple Integral

Definition

Let $Q = [a,b]\times[c,d]\times[r,s]$ be a cube in the space, and $f\colon Q \to \mathbb{R}$ be a function.
f is said to be Riemann integrable on Q if there exists a real number I such that
$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that if $\mathcal{P}$ is partition of Q satisfying $\|\mathcal{P}\| < \delta$, then any
Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(I-\varepsilon, I+\varepsilon)$.
Such a number I is called Riemann integral or triple integral of f on Q
and is denoted by $\displaystyle\iiint_Q f(x,y,z)\,dV$.

Riemann Integral 存在則唯一！

Fubini's Theorem

Let $B = [a,b]\times[c,d]\times[r,s]$. If f satisfies $\begin{cases} f \text{ is continuous on } B \\ f \text{ is bounded on } B \text{ and is discontinuous only at a finite number of smooth surfaces'} \end{cases}$

$$\begin{aligned}\text{then } \iiint_B f(x,y,z)\,dV &= \int_a^b\int_c^d\int_r^s f(x,y,z)\,dzdydx = \int_a^b\int_r^s\int_c^d f(x,y,z)\,dzdxdy \\ &= \int_c^d\int_a^b\int_r^s f(x,y,z)\,dydzdx = \int_c^d\int_r^s\int_a^b f(x,y,z)\,dydxdz \\ &= \int_r^s\int_c^d\int_a^b f(x,y,z)\,dxdydz = \int_r^s\int_a^b\int_c^d f(x,y,z)\,dxdzdy.\end{aligned}$$

Triple Integrals over General Region

Type 1

Let $E = \{(x,y,z)\,|\,a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdydx$.

Type 2

Let $E = \{(x,y,z)\,|\,c \le y \le d,\ g_1(y) \le x \le g_2(y),\ h_1(x,y) \le z \le h_2(x,y)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_c^d\int_{g_1(y)}^{g_2(y)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdxdy$.

Type 3

Let $E = \{(x,y,z)\,|\,c \le y \le d,\ g_1(y) \le z \le g_2(y),\ h_1(y,z) \le x \le h_2(y,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_c^d\int_{g_1(y)}^{g_2(y)}\int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dxdzdy$.

Type 4

Let $E = \{(x,y,z)\,|\,r \le z \le s,\ g_1(z) \le y \le g_2(z),\ h_1(y,z) \le x \le h_2(y,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_r^s\int_{g_1(z)}^{g_2(z)}\int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dxdydz$.

Type 5

Let $E = \{(x,y,z)\,|\,a \le x \le b,\ g_1(x) \le z \le g_2(x),\ h_1(x,z) \le y \le h_2(x,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dydzdx$.

Type 6

Let $E = \{(x,y,z)\,|\,r \le z \le s,\ g_1(z) \le x \le g_2(z),\ h_1(x,z) \le y \le h_2(x,z)\}$.
Then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_r^s\int_{g_1(z)}^{g_2(z)}\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dydxdz$.

Substitution in Multiple Integrals

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現在要來介紹在重積分時，變數變換的作法！

Jacobian

The jacobian of the transformation T given by $x = g(u,v,w)$ and $y = h(u,v,w)$ and $z = k(u,v,w)$ is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Jacobian 在幾何上可以看成是，變數變換後的區域微小面積（或體積）與原來區域微小面積（或體積）之間的比例因子。

Theorem

Suppose that $\begin{cases} T \text{ is a } C^1 \text{ transformation (即 All partial derivatives are continuous).} \\ \text{Jacobian of } T \text{ doesn't equal 0.} \\ T \text{ is one-to-one (即反函數 } T^{-1} \text{ 存在), except perhaps on the boundary of S.} \end{cases}$ holds,

f is continuous on R, and T maps region S in the $uvw-$space onto the region R in $xyz-$space.

Then $\displaystyle\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \iiint_S f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\det(J)\right| du\,dv\,dw$.

Cylindrical coordinates

坐標轉換

柱坐標→直角坐標 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

直角坐標→柱坐標 $\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$

計算公式

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Spherical coordinates

坐標轉換

球坐標→直角坐標 $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$

直角坐標→球坐標 $\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\theta = \frac{y}{x} \\ \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$

計算公式

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$