多變數純量函數偏導數（應用篇）

算近似值 — Linear Approximation

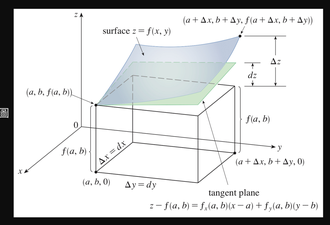
之前有提到可微則存在切平面，我們可以用切平面作線性近似，說明如下。

If f is differentiable at (a,b),
then the approximating function $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is the linearization of f at (a,b).
The approximation $f(x,y) \approx L(x,y)$ is called the linear approximation of f at (a,b).

Total Differential

For a differentiable function of two variables, $z = f(x,y)$,
we define the differentials dx and dy to be independent variables; that is, they can be given any values.
Then the differential dz, also called the total differential, is defined by $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$.

示意圖



Application of Gradient

Relationship with Directional Derivative

Let f be a differentiable function. $\mathbf{D_u} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.

If $\nabla f(\mathbf{x}) \neq \mathbf{0}$: 1. $\nabla f(\mathbf{x})$ points in the direction of maximum rate of change of f at $\mathbf{x}$, while -$\nabla f(\mathbf{x})$ points in the direction of minimum rate of change of f at $\mathbf{x}$. 2. The maximum rate of change is $|\nabla f(\mathbf{x})|$, and the minimum rate of change is -$|\nabla f(\mathbf{x})|$.

If $\nabla f(\mathbf{x}) = \mathbf{0}$: $D_u f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{u}$.

這些都是要在可微的情況下才會發生！

Level Curve and Level Surface（Salas 的書都有解釋，可以去看！）

梯度垂直等高線和等高面

At each point of the domain, the gradient vector ∇f, if not $\mathbf{0}$, is prependicular to the level curve or level surface of f through that point.

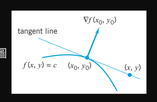
⚠ Salas 的書有證明，可以去翻翻看！

Level Curve $f(x,y) = c$

Tangent Line

Equation $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$

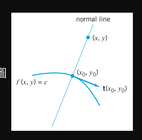
示意圖



Normal Line

Equation $f_y(x_0,y_0)(x-x_0) - f_x(x_0,y_0)(y-y_0) = 0$

示意圖

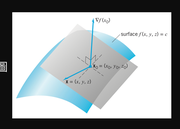


Level Surface $f(x,y,z) = c$

Tangent Plane

Equation $f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$

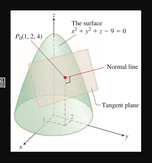
示意圖



Normal Line

Equation $x = x_0 + f_x(x_0,y_0,z_0)\,t$
$y = y_0 + f_y(x_0,y_0,z_0)\,t$
$z = z_0 + f_z(x_0,y_0,z_0)\,t$

示意圖



補充：找雙變數函數 Graph 的 Tangent Plane 和 Normal Line 的方法

A surface of the form $z = g(x,y)$ can be written in the form $f(x,y,z) = 0$ by setting $f(x,y,z) = g(x,y) - z$. If g is differentiable, so is f. Moreover, $\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = \frac{\partial g}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y,z) = \frac{\partial g}{\partial y}(x,y) \\ \frac{\partial f}{\partial z}(x,y,z) = -1 \end{cases}$.

Tangent Plane

The tangent plane at (x_0,y_0,z_0) has equation $g_x(x_0,y_0)\,(x-x_0) + g_y(x_0,y_0)\,(y-y_0) + (-1)(z-z_0) = 0$, which we can rewrite as $z - z_0 = g_x(x_0,y_0)\,(x-x_0) + g_y(x_0,y_0)\,(y-y_0)$. If $\nabla g(x_0,y_0) = \mathbf{0}$, then both partials of g are zero at (x_0,y_0) and the equation reduces to $z = z_0$. In this case, the tangent plane is horizontal.

Normal Line

$x = x_0 + g_x(x_0,y_0)\,t$
$y = y_0 + g_y(x_0,y_0)\,t$
$z = z_0 + (-1)\,t$

極值介紹

Absolute Extrema 絕對極值

Suppose that f is a function of several variables.

The function f is said to have a $\begin{cases} \text{absolute maximum} \\ \text{absolute minimum} \end{cases}$ at $\mathbf{x}_0$ provided that $\begin{cases} f(\mathbf{x}_0) \geq f(\mathbf{x}) \\ f(\mathbf{x}_0) \leq f(\mathbf{x}) \end{cases}$
for all $\mathbf{x}$ in the domain of f. The absolute maxima and minima of f comprise the absolute extreme values of f.

Local Extrema 局部極值

Suppose that f is a function of several variables and $\mathbf{x}_0$ is an interior point of the domain.

The function f is said to have a $\begin{cases} \text{local maximum} \\ \text{local minimum} \end{cases}$ at $\mathbf{x}_0$ provided that $\begin{cases} f(\mathbf{x}_0) \geq f(\mathbf{x}) \\ f(\mathbf{x}_0) \leq f(\mathbf{x}) \end{cases}$
for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$. The local maxima and minima of f comprise the local extreme values of f.

找極值

找相對極值

Theorem

If f has a local maximum or minimum at $\mathbf{x}_0$, then $\mathbf{x}_0$ is a critical point of f.

Critical Points

Let f defined on an open region R containing (x_0,y_0).
The point $\mathbf{x}_0$ is a critical point of f if one of the following is true.
1. $\nabla f(\mathbf{x}_0) = \mathbf{0}$.
2. $\nabla f(\mathbf{x}_0)$ does not exist.
我們稱 Critical Points 中，$\nabla f(\mathbf{x}_0) = \mathbf{0}$ 的那些 $\mathbf{x}_0$ 為 Stationary Points。
如果一個 Stationary Points 沒有極值，就稱它為 Saddle Point（馬鞍點）。

找相對極值方法 Second Derivatives Test

Let f have continuous second partial derivatives on an open region containing a point (a,b) and $\nabla f(a,b) = \langle 0,0 \rangle$.
Consider the matrix $\begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$ and the quantity $D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(1) If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b) .
(2) If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b) .
(3) If $D < 0$, then $(a,b,f(a,b))$ is a saddle point.
(4) If $D = 0$, then the test fails.
⚠ 想知道更多，可以去看陽交大蘇承芳老師的講義！

跟單變數的時候一樣，為了知道 Critical Points 有沒有相對極值，我們需要 Second Derivatives Test 來告訴我們，畢竟不是所有的 Critical Points 都有相對極值！

找絕對極值

Extreme Value Theorem

If f is continuous on a closed and bounded set D, then on that set f takes on an absolute maximum and an absolute minimum.

Bounded

A subset of the plane or three-space is said to be bounded provided there exists a positive number R such that $\|\mathbf{x}\| \leq R$ for all $\mathbf{x} \in S$.

找絕對極值方法

To find the absolute extrema of a continuous function f on a closed, bounded set D:
1. Find the value of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. Compare the values of 1 and 2.（這些值中最大的就是絕對極大值；最小的就是絕對極小值！）

拉格朗日乘數法（適用於有條件的情況下）

在一個限制下

定理

Let f and g have continuous first partial derivatives such that
f has an extremum at a point (x_0,y_0,z_0) on the smooth constraint curve $g(x,y,z) = c$.
If $\nabla g(x_0,y_0,z_0) \neq \langle 0,0,0 \rangle$, then there is a real number λ such that $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$.

計算方法

(1) Find all value of x, y, z, and λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = c$.
(2)將上面所得到的所有的 x,y,z 全部代入 $f(x,y,z)$ 裡把值算出來。這些值中最大的就是 f 的最大值，最小的就是 f 的最小值。

在兩個限制下

定理

If $f(x,y,z)$ subject to the side constraints $g(x,y,z) = k$ and $h(x,y,z) = c$ and f has an extremum at a point (x_0,y_0,z_0),
then $\nabla f(x_0,y_0,z_0)$ is in the plane determined by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$.
If $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$ are not parallel and not $\langle 0,0,0 \rangle$, then there exist λ and μ such that $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$.

計算方法

(1) Find all value of x, y, z, λ, and μ such that $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = c \\ h(x,y,z) = k \end{cases}$
(2)將上面所得到的所有的 x,y,z 全部代入 $f(x,y,z)$ 裡把值算出來。這些值中最大的就是 f 的最大值，最小的就是 f 的最小值。