多變數純量函數偏導數（概念篇）

Partial Derivative 偏導數

概念

偏導數是一種只針對多變數函數某一個變數微分而得的導數，且微分的過程裡將其他的變數都視為常數。
我們從前學習單變數函數時沒有偏導數的概念，但在多變數函數的世界裡，偏導數是個很重要的工具。
為了簡化我們的學習，我們先從雙變數函數 $z=f(x,y)$ 開始考慮，只要同學們充分了解 $z=f(x,y)$ 的偏導數如何運算，推廣到 $y=f(x_1,x_2,\cdots,x_n)$ 做法也會相同！

Definition

If u is a function of n variables, $u=f(x_1,x_2\cdots,x_n)$, its partial derivative with respect to the ith variable x_i is $\dfrac{\partial u}{\partial x_i}=\lim\limits_{h\to 0}\dfrac{f(x_1,\ldots,x_i+h,\ldots,x_n)-f(x_1,\ldots,x_n)}{h}$ and we also write $\dfrac{\partial u}{\partial x_i}=\dfrac{\partial f}{\partial x_i}=f_{x_i}=D_{x_i}f$.

範例：雙變數函數偏導數

If f is a function of two variables, its partial derivatives are the function f_x and f_y defined by

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

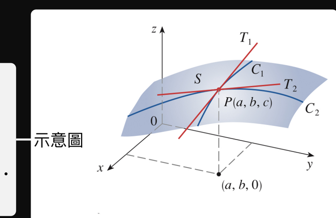
$$f_y(x,y)=\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{h}$$

幾何意義

Let $\begin{cases}(1)\text{ The surface } S \text{ be the graph of } f(x,y)\\(2)\text{ The curve } C_1 \text{ be intersection of the plane } y=b \text{ and } S\\(3)\text{ The curve } C_2 \text{ be intersection of the plane } x=a \text{ and } S\end{cases}$

Suppose that $f(a,b)=c$ then $\begin{cases}f_x(a,b)\text{ is the slope (斜率) of the tangent line at } (a,b,c) \text{ to } C_1\,.\\f_y(a,b)\text{ is the slope of the tangent line at } (a,b,c) \text{ to } C_2\,.\end{cases}$

示意圖



The partial derivatives of f at (a,b) are the slopes of the tangents to C_1 and C_2.

Partial Derivative & Continuity

雖然單變數函數導數存在保證連續，
但是多變數函數偏導數存在無法保證連續！

例子（以雙變數函數為例）：
在原點偏導數存在，
但極限不存在，
所以一定不連續。

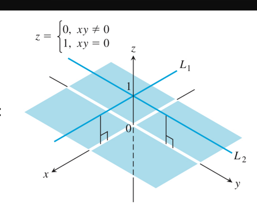


FIGURE 14.21 The graph of $f(x,y)=\begin{cases}0, & xy\neq 0\\1, & xy=0\end{cases}$

Higher Derivative（以二階偏導數為例）

Let $z=f(x,y)$.

Differentiate twice w.r.t x: $\dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial x})=\dfrac{\partial^2 f}{\partial x^2}=f_{xx}=(f_x)_x=f_{11}$

Differentiate twice w.r.t y: $\dfrac{\partial}{\partial y}(\dfrac{\partial f}{\partial y})=\dfrac{\partial^2 f}{\partial y^2}=f_{yy}=(f_y)_y=f_{22}$

Differentiate first w.r.t x and then w.r.t y: $\dfrac{\partial}{\partial y}(\dfrac{\partial f}{\partial x})=\dfrac{\partial^2 f}{\partial y\partial x}=f_{xy}=(f_x)_y=f_{12}$

Differentiate first w.r.t y and then w.r.t x: $\dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial y})=\dfrac{\partial^2 f}{\partial x\partial y}=f_{yx}=(f_y)_x=f_{21}$

Proposition — Partial Differentiation Rules

$$\begin{cases}(cf)_{x_i}=cf_{x_i} \quad c\in\mathbb{R}\\(f+g)_{x_i}=f_{x_i}+g_{x_i}\\(f\cdot g)_{x_i}=f_{x_i}g+g_{x_i}f\\(\dfrac{f}{g})_{x_i}=\dfrac{f_{x_i}g-g_{x_i}f}{g^2}\end{cases}$$

Theorem — Clairaut's Theorem

Let $z=f(x,y)$. If f_{xy} and f_{yx} are continuous on an open disk D then $f_{xy}(x,y)=f_{yx}(x,y) \quad \forall\,(x,y)\in D$.

多變數函數可微

我們通常不會用定義去看一個點可不可微，我們會用下面的定理去看！（以雙變數函數為例）
Let $z=f(x,y)$ be a function of two variables.
If the partial derivatives f_x and f_y exists near (a,b) and are continuous at (a,b),
then f is differentiable at (a,b).

Definition

Denote $\begin{cases}\text{the increment of } x \text{ by } \Delta x\\\text{the increment of } y \text{ by } \Delta y\\\text{the increment of } z \text{ by } \Delta z\end{cases}$, so $\Delta z=f(a+\Delta x,b+\Delta y)-f(a,b)$.

Then, $f(x,y)$ is differentiable at (a,b) if $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$, where $\varepsilon_1,\varepsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$.

Tangent Plane

If $z=f(x,y)$ is differentiable at (x_0,y_0), then there exists a tangent plane at the point (x_0,y_0,z_0),
and the equation of the tangent plane is $z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$.

補充：Scalar parametric equations for the line normal to the tangent plane at the point (x_0,y_0,z_0) is $\begin{cases}x=x_0+f_x(x_0,y_0)\,t\\y=y_0+f_y(x_0,y_0)\,t\\z=z_0+(-1)\,t\end{cases}$.

Proposition

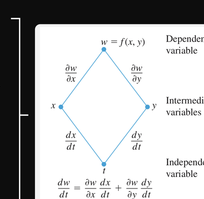
可微必連續 If f is differentiable at $\mathbf{x}$, then f is continuous at $\mathbf{x}$.

Chain Rule

簡寫符號說明：
(2,1) 表示兩個 Intermediate Variables 和一個 Independent Variable

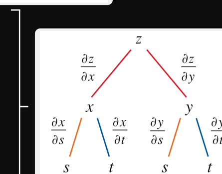
Case 1: (2,1)

Suppose $w=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable function of t.
Then w is a differentiable function of t, and $\dfrac{dw}{dt}=\dfrac{\partial w}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$.



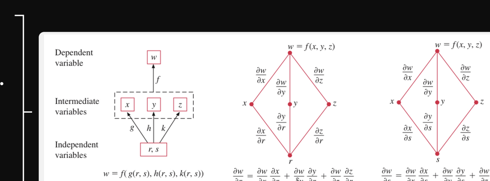
Case 2: (2,2)

Suppose $z=f(x,y)$ is a differentiable function of x and y, where $x=g(s,t)$ and $y=h(s,t)$ are both differentiable function of s and t.
Then, $\dfrac{\partial z}{\partial s}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ $\quad \dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$.



Case 3: (3,2)

Suppose $w=f(x,y,z)$ is a differentiable function of x and y and z, where $x=g(r,s)$ and $y=h(r,s)$ and $z=j(r,s)$ are both differentiable function of r and s.
Then $\dfrac{\partial w}{\partial r}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial w}{\partial s}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial s}$



General Version

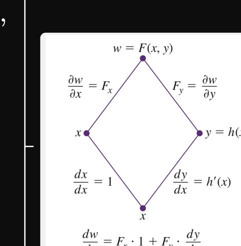
Suppose that u is a differentiable function of the n variables $x_1,x_2,\ldots,x_n$ and each x_j is a differentiable function of the m variables $t_1,t_2,\ldots,t_m$
and $\dfrac{\partial u}{\partial t_i}=\dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial t_i}+\dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial t_i}+\cdots+\dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}$ for each $i=1,2,\ldots,m$.

Implicit Differentiation

之前有提過，但那時無法解釋，現在用雙變數函數當例子，回憶一下！
Suppose that $w=F(x,y)$ is a continuously differentiable function. If y is a differentiable function of x that satisfies $w=0$, then we can find the derivative of y respect to x without first having to express y explicitly in terms of x.
The process by which we do this is called implicit differentiation.

雙變數函數

According to the Chain Rule for functions of one independent variable and two intermediate variables,
suppose that $\begin{cases}\text{The functions } F(x,y) \text{ is differentiable.}\\\text{The equation } F(x,y)=0 \text{ defines } y \text{ implicitly as a differentiable function of } x.\end{cases}$
Since $w=F(x,y)=0$, the derivative $\dfrac{dw}{dx}$ must be 0. Computing the derivative from the chain rule,
we find $0=\dfrac{dw}{dx}=F_x\dfrac{dx}{dx}+F_y\dfrac{dy}{dx}=F_x\cdot 1+F_y\dfrac{dy}{dx}$.
If $F_y=\dfrac{\partial w}{\partial y}\neq 0$, we can solve this to get $\dfrac{dy}{dx}=-\dfrac{F_x(x,y)}{F_y(x,y)}$.



三變數函數 If the equation $F(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x(x,y,z)}{F_z(x,y,z)}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y(x,y,z)}{F_z(x,y,z)}$, $F_z(x,y,z)\neq 0$.

Directional Derivative

Definition

For each unit vector (單位向量) $\mathbf{u}$, the directional derivative (方向導數) of f at $\mathbf{x}$ in the direction of $\mathbf{u}$ is
$D_{\mathbf{u}}f(\mathbf{x})=\lim\limits_{h\to 0}\dfrac{f(\mathbf{x}+h\mathbf{u})-f(\mathbf{x})}{h}$ if the limit exists.
⚠ 偏導數可看成是方向導數的特例。

Directional Derivative & Continuity

Let f be the scalar function defined on $\mathbb{R}^2$ as follows: $f(x,y)=\dfrac{xy^2}{x^2+y^4}$ if $x\neq 0$, $f(0,y)=0$.
Let $\mathbf{c}=(0,0)$ and let $\mathbf{y}=(a,b)$ be any vector. If $a\neq 0$ and $h\neq 0$, we have
$\lim\limits_{h\to 0}\dfrac{f(\mathbf{c}+h\mathbf{y})-f(\mathbf{c})}{h}=\lim\limits_{h\to 0}\dfrac{ab^2}{a^2+h^2b^4}=\dfrac{b^2}{a}$. Therefore, the direcrtional derivative exists
for all direction and equals 0. However, at each point of the parabola $x=y^2$ (except the origin)
the function has the value $\dfrac{1}{2}$. Thus, the function is not continuous at $\mathbf{c}$.

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Theorem If f is a differentiable at $\mathbf{x}$, then $\begin{cases}f \text{ has a directional derivative at } \mathbf{x} \text{ in every direction.}\\\text{for each unit vector } \mathbf{u},\ \mathbf{D_u}f(\mathbf{x})=\langle f_x(x,y),f_y(x,y)\rangle\cdot\mathbf{u}.\end{cases}$

Gradient

Definition

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by $\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle$

Differentiation Rules

Suppose that α is a constant and f and g are both differentiable.

$$\nabla[f(\mathbf{x})+g(\mathbf{x})]=\nabla f(\mathbf{x})+\nabla g(\mathbf{x})$$

$$\nabla[\alpha f(\mathbf{x})]=\alpha\nabla f(\mathbf{x})$$

$$\nabla[f(\mathbf{x})g(\mathbf{x})]=f(\mathbf{x})\nabla g(\mathbf{x})+g(\mathbf{x})\nabla f(\mathbf{x})$$

Theorem

Mean-Value Theorem: If f is differentiable at each point of line segment $\overline{\mathbf{ab}}$, then there exists on that line segment a point $\mathbf{c}$ between $\mathbf{a}$ and $\mathbf{b}$ such that $f(\mathbf{b})-f(\mathbf{a})=\nabla f(\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})$.

Let U be an open connected set and let f be a differentiable function on U.
If $\nabla f(\mathbf{x})=\mathbf{0}$ for all $\mathbf{x}$ in U, then f is constant on U.

Corollary: Let U be an open connected set and let f and g be functions differentiable on U.
If $\nabla f(\mathbf{x})=\nabla g(\mathbf{x})$, then f and g differ by a constant on U.

補充：對位置函數取梯度公式

Let $r(x,y,z)=\sqrt{x^2+y^2+z^2}$; $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$.
$\nabla r^n=nr^{n-2}\mathbf{r} \quad \forall\, n\in\mathbb{Z},\mathbf{r}\neq 0$
If n is positive and even, the results also holds at $\mathbf{r}=0$.

關係整理圖

Let $z=f(x,y)$ be a function of two variable.

f 在 (a,b) 上的方向導數都存在
且 $D_{\vec u}f(a,b)=\nabla f(a,b)\cdot\vec u$

可微則偏導數必存在

f_x, f_y 皆存在 near (a,b)

+ f_x, f_y are conti. at (a,b) $\Longrightarrow$ f is diff. at (a,b)

可微必連續

+ f_x, f_y are bonded near (a,b) $\Longrightarrow$ f is conti. at (a,b)