單變數向量函數（應用篇）

Curve

Definition

A subset C in the plane (or space) is called a curve if C is the image of an interval $I \subseteq \mathbb{R}$ under a continuous vector function $\mathbf{r}$.

The continuous function $\mathbf{r}\colon I \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a parametrization of the curve, and the equations $\begin{cases} x = r_1(t) \\ y = r_2(t),\ t \in I\text{（以三維為例）}, \\ z = r_3(t) \end{cases}$

where x, y, and z are simply scalar functions of t, are called parametric equations of the curve, and t is called a parameter. ⚠ 參數化方式不唯一！

- **Simple**: A curve C is called simple if it has an injective parametrization; that is, there exists $\mathbf{r}\colon I \to \mathbb{R}^3$ such that $\mathbf{r}(I) = C$ and $\mathbf{r}(x) = \mathbf{r}(y)$ implies that $x = y$.
  - 就是這個函數是一對一的意思。
- **Closed**: A curve C with parametrization $\mathbf{r}\colon I \to \mathbb{R}^3$ is called closed if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\mathbf{r}(a) = \mathbf{r}(b)$.
- **Smooth**: A smooth curve is a curve with differentiable parametrization $\mathbf{r}\colon I \to \mathbb{R}^3$ such that $\mathbf{r}'(t) \neq 0$ for all $t \in I$.
  - 就是整個函數都可以微分的意思且微分不為 0。

Arc Length

計算方式

Let C be a curve parametrized by an injective continuously differentiable parametrization $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$.

Then length of C is $L = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$.

- 其實這也蠻直觀的，將曲線上每一個微小的切向量的長度相加，當這些微小切向量的長度之和在取極限時，自然就得到了曲線的長度。這裡的積分本質上就是在處理這樣的極限情況。

Arc Length Function

Definition

The arc length function s for a curve given by a vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \quad a \le t \le b$,

is $s(t) = \int_\alpha^t \|\mathbf{r}'(u)\|\, du = \int_\alpha^t \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2 + (\frac{dz}{du})^2}\, du.$

補充：By the fundamental Theorem of Calculus, $\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}.$

Then by Substitution Rule for Definite Integrals, $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt = \int_a^b ds.$

最後一個式子告訴我們把所有微量長度相加，直觀來看，會得到整條曲線的長度！

弧長參數

Since $\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0$, the function $s(t)$ is a one-to-one increasing function.
Thus, no two points of C can lie at the same arc distance from $\mathbf{r}(a)$. It follows that for each $s \in [0, L]$, there is a unique point $\mathbf{R}(s)$ on C at arc distance s from $\mathbf{r}(a)$.
The vector function $\mathbf{R}(s),\ s \in [0, L]$ parametrizes C by arc length. We call s the arc length parameter for the curve.

補充 1：Since $\frac{ds}{dt} = \left\| \frac{d\mathbf{r}}{dt} \right\| > 0$ the function $s(t)$ has a differentiable inverse $t(s)$ and we can write

$\frac{dt}{ds} = \frac{1}{\|\frac{d\mathbf{r}}{dt}\|}$（套導數單元教過的反函數公式）. Since both $\mathbf{R}(s(t))$ and $\mathbf{r}(t)$ lies at arc distance $s(t)$ from $\mathbf{r}(a)$,

$\mathbf{R}(s(t)) = \mathbf{r}(t)$. Therefore $\mathbf{R}(s) = \mathbf{r}(t(s))$. Differentiation gives $\frac{d\mathbf{R}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{d\mathbf{r}}{dt}\frac{1}{\|\frac{d\mathbf{r}}{dt}\|}$.

Taking the norm of both sides, we have $\frac{d\mathbf{R}}{ds} = 1$. Thus, for a curve parametrized by arc length, the tangent vector can change in direction but not in length: the tangent vector maintains length 1.

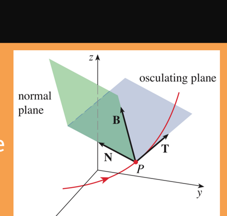
補充 2：Moreover, if the curve C: $\mathbf{r}(t),\ t \in [0.b]$ has tangent vector of constant length 1, then the parametrization is by arc length and the length of the curve is b.

- **範例**: Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from (1,0,0) in the direction of increasing t.

  $\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{2} \Rightarrow s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t.$

  Therefore $t = \frac{s}{\sqrt{2}}$ and the required reparametrization is obtained by substituting for t: $\mathbf{r}(t(s)) = \cos\frac{s}{\sqrt{2}}\,\mathbf{i} + \sin\frac{s}{\sqrt{2}}\,\mathbf{j} + \frac{s}{\sqrt{2}}\,\mathbf{k} \quad s \ge 0$

TNB Frame



Unit Tangent Vector

Let C be a smooth curve represented by $\mathbf{r}$ on an interval I.
The unit tangent vector $\mathbf{T}$ is defined as $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

Curvature

直觀看法

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of $\mathbf{T}$.

定義

Let C be a smooth curve given by $\mathbf{r}(s)$, where s is the arc length parameter.
The curvature κ at s is $\kappa = \|\frac{d\mathbf{T}}{ds}\| = \|\mathbf{T}'(s)\|$ where $\mathbf{T}$ is the unit tangent vector.

- **計算方式**: The curvature is easier to compute if it is expressed in terms of the parameter t instead of s so we use Chain Rule to write $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \Rightarrow \kappa = \|\frac{d\mathbf{T}}{ds}\| = \|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\|$.
  And $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, so we can also write $\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.

Theorem

If C is a smooth curve given by $\mathbf{r}(t)$,
then the curvature κ of C at t is $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.

Theorem

Suppose that C is twice differentiable. If C has nonzero tangent vector $\mathbf{r}'(t) = x'(t)\,\mathbf{i} + \mathbf{y}'(t)\,\mathbf{j}$, then $\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{([x'(t)]^2 + [y'(t)]^2)^{\frac{3}{2}}}$.

- For the graph of a twice differentiable function $y = f(x)$, $\kappa = \frac{|y''(x)|}{(1 + [y'(x)]^2)^{\frac{3}{2}}}$.

Principal Unit Normal

概念

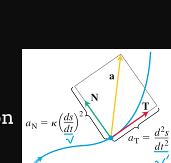
Among the vectors orthogonal to the unit tangent vector $\mathbf{T}$, there is one of particular significance because it points the direction in which the curve is turning. Since $\mathbf{T}$ has constant length, the derivative $\frac{d\mathbf{T}}{ds}$ is orthogonal to $\mathbf{T}$. Therefore, if we divide $\frac{d\mathbf{T}}{ds}$ by its length, we obtain a unit vector $\mathbf{N}$ orthogonal to $\mathbf{T}$.

定義

At a point where $\|\mathbf{T}'(s)\| \neq \mathbf{0}$, the principal unit normal vector for a smooth curve in the plane is $\mathbf{N} = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$.

- **其它計算公式**: If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, we can use the Chain Rule to calculate N directly:
  $\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{\frac{d\mathbf{T}}{dt}\frac{dt}{ds}}{|\frac{d\mathbf{T}}{dt}||\frac{dt}{ds}|} = \frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{dt}$.

Tangential and Normal Components of Acceleration



概念

When we study the motion of a partical, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal.
If we write $v = |\mathbf{v}|$ for the speed of the partical,

then $\mathbf{T}(t) = \frac{\frac{d\mathbf{r}}{dt}}{\|\frac{d\mathbf{r}}{dt}\|} = \frac{\mathbf{v}(t)}{v(t)}$ and so $\mathbf{v} = v\mathbf{T}$.

Differentiation with respect to t gives $\mathbf{a} = v'\mathbf{T} + v\mathbf{T}'$.

$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \Rightarrow \mathbf{T}' = \|\mathbf{T}'\|\mathbf{N} = \kappa v \mathbf{N} \quad (\|\mathbf{T}'\| = \kappa\|\mathbf{r}'(t)\| = \kappa v)$

Finally, we have $\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$.
從這個式子可以知道，加速度一定在 $\mathbf{T}$ 和 $\mathbf{N}$ 所構成的平面上！

公式

If the acceleration vector is written as $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$,

then $\begin{cases} a_T = \frac{d^2s}{dt^2} = \frac{dv}{dt} = v' \\ a_N = \kappa(\frac{ds}{dt})^2 = \kappa v^2 \end{cases}$ are the tangential and normal scalar components of acceleration.

- $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a} = (a_T)^2 + (a_N)^2 \Rightarrow a_N = \sqrt{\|\mathbf{a}\| - (a_T)^2}$
  With this formula, we can find a_N without having to calculate κ first.

Binormal Vector

概念

Binormal vector is the tendency of your motion to "twist" out the plane created by $\mathbf{T}$ and $\mathbf{N}$ in the direction perpendicular to this plane. 用中文說就是，
$\mathbf{B}$ 是對於從 $\mathbf{T}$ 和 $\mathbf{N}$ 創造的平面上「扭曲」到與該平面垂直的方向的運動傾向！

定義 $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

- The vector $\mathbf{B}$ is defined so that the ordered triple $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed system. And $\mathbf{B}$ is a unit vector since $\|\mathbf{B}\| = \|\mathbf{T}\|\,\|\mathbf{N}\| \sin\frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1.$

Torsion

概念

A space curve can also lift or "twist" out of the osculating plane at P.
Since $\mathbf{B}$ is normal to the osculating plane, $\frac{d\mathbf{B}}{ds}$ gives us information about how the osculating plane changes as P moves along C.
Since $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{B}$ (the latter has constant length) and $\mathbf{T}$,
It follows that $\frac{d\mathbf{B}}{ds}$ is orthogonal to the plane $\mathbf{B}$ and $\mathbf{T}$. In other words,
$\frac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, so $\frac{d\mathbf{B}}{ds}$ is a scalar multiple of $\mathbf{N}$. In symbols,
$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. (The negative sign in this equation is traditional.)
τ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve.

- **補充**: Prove that $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$.
  $\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T} \times \mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \quad$ ($\mathbf{N}$ is the direction of $\frac{d\mathbf{T}}{ds}$)
  $= \mathbf{T} \times \frac{d\mathbf{N}}{ds}$

定義

$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau$

$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$

- Torsion is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write $\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt}$ so $\frac{d\mathbf{B}}{ds} = \frac{\frac{d\mathbf{B}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{B}'(t)}{\|\mathbf{r}'(t)\|}$.
  Finallly, we have $\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|}$.

Theorem

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\|\mathbf{v} \times \mathbf{a}\|^2}$$

Frenet-Serret Formulas

$\mathbf{T}'(s) = \kappa\mathbf{N}$
$\mathbf{N}'(s) = -\kappa\mathbf{T} + \tau\mathbf{B}$
$\mathbf{B}'(s) = -\tau\mathbf{N}$
想看證明可以去看 Vector Calculus by Susan Jane Colley。