```
Definition A function f \colon \mathbb{N} 	o \mathbb{R} 	ext{ is called a sequence.}
                                                                 A sequence \{a_n\} has the limit L and we write
                                                                    \lim_{n	o\infty} a_n = L \quad or \quad a_n 	o L \ as \ n 	o \infty
                                                                 if for every \varepsilon > 0 there is a corresponding interger N such that
                                                                                                                                                     若極限值存在則唯一
                                              Limit of a Sequence
                                                                   	ext{if} \quad n>N \quad 	ext{then} \quad |a_n-L|<arepsilon
                                                                                 that has a limit is said to be convergent.
                                                           The notation \lim a_n = \infty means that for every M > 0
                                              Infinite Limit
                                                           there is an interger N such that if n > N then a_n > M
                                                           If \lim_{x \to \infty} f(x) = L and f(n) = a_n for all n \in \mathbb{N}, then \lim_{x \to \infty} a_n = L. L'Hospital Rule 不能用在數列的極限上,但可以用在函數的極限上,再引用得到數列的極限
                                                                                                                                                               \lim_{n	o\infty}(a_n\pm b_n)=\lim_{n	o\infty}a_n\pm\lim_{n	o\infty}b_n
                        nfinite Sequence
                                                                                                                                                                \lim ca_n = c \lim a_n
                                                                                        Suppose that \{a_n\} and \{b_n\} are convergent sequences
                                                                                                                                                               \lim_{n	o\infty}(a_nb_n)=\lim_{n	o\infty}a_n\lim_{n	o\infty}b_n
                                                           − 🕞 Limit Laws for Sequences
                                                                                        and c is a constant.
                                                                                                                                                              -\lim_{n	o\infty}a_n{}^p=\left[\lim_{n	o\infty}a_n
ight]^p\quad	ext{if}\ \ p>0\ \ 	ext{and}\ \ a_n>0
                                              -Proposition •
                                                           If \lim_{n\to\infty}|a_n|=0, then \lim_{n\to\infty}a_n=0.
                                                            \text{ If } \lim_{n \to \infty} a_n = L \text{ and the function } f \text{ is continuous at } L, \text{ then } \lim_{n \to \infty} \overline{f(a_n)} = \overline{f(L)}. 
                                                           lacksquare Squeeze Theorem 	ext{If } a_n \leq b_n \leq c_n 	ext{ for } n \geq n_0 	ext{ and } \lim_{n 	o \infty} a_n = \lim_{n 	o \infty} c_n = L, 	ext{ then } \lim_{n 	o \infty} b_n = L.
                                                                                  -Monotonic 定義 A sequence is called monotonic if it is either increasing or decreasing.
                                                                                  -Bounded 定義 If a sequence is bounded above or below, then it is called a bounded sequence.
                                              - Monotonic and Bounded Sequence
                                                                                                         Every bounded, monotonic sequence is convergent.
                                                                               Given a series \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots, let s_n denote its nth partial sum :
                                         Partial Sum
                                                                           s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n
                                                                    If the sequence \{s_n\} is convergent and \lim_{n\to\infty}s_n=s\in\mathbb{R},
                                                                    then the series is called convergent and we write \sum a_n = s ( The sum of the series )
                                         一級數的斂散性一
數列與級數
                                                       Divergent If the sequence \{s_n\} is divergent, then the series is called divergent.
                                                                      If |r| \geq 1, the geometric series is divergent.
                                                    Geometric Series If |r| < 1, the geometric series is convergent and its sum is \sum_{1}^{\infty} ar^{n-1} = \frac{a}{1-r}
                                         – Example
                                                       \sum_{n \to \infty} a_n is convergent, then \lim_{n \to \infty} a_n = 0.
                                         - Proposition -
                                                      igcap_{\oplus} 無窮級數運算 Suppose that \sum a_n and \sum b_n are convergent series and c is a constant.
                                                                                                                                                                                                                                             If R_n = s - s_n then
                                                                                    Integral Test
                                                                                                                                                                                                                                            s_n + \int_{-1}^\infty f(x) \, dx \leq s \leq s_n + \int_{-1}^\infty f(x) \, dx
                                                                                                 If \int_{1}^{\infty} f(x) dx is divergent, then \sum_{n=1}^{\infty} a_n is divergent.
                                                                                                                  If |r| \geq 1, the geometric series is divergent.
                                                                                                Geometric Series If |r| < 1, the geometric series is convergent and its sum is \sum_{1}^{\infty} ar^{n-1} = \frac{a}{1-r}
                                                                                               p-series The p-series \sum_{n=1}^{\infty} \frac{1}{n^p} is convergent if p>1 and divergent if p\leq 1.
                                                                                                                                      If \sum b_n is convergent and a_n \leq b_n for all n, then \sum a_n is also convergent.
                                                                                                          (收斂的話可以更進一步預估和) If \sum b_n is divergent and a_n \geq b_n for all n, then \sum a_n is also divergent.
                                                                                     Comparison Tests
                                                                                                       Limit Comparison Test If \lim_{n \to \infty} \frac{a_n}{b_n} = c where c > 0, then either both series converge or both diverge.
                                                                                        If the alternating series \sum_{n=0}^{\infty} (-1)^{n-1}b_n = b_1-b_2+b_3-b_4+\cdots \ (b_n>0)
                                                                                                                                                                                                                                   If s = \sum_{n=0}^{\infty} (-1)^{n-1} b_n, where b_n > 0, is the sum of an alternating series that satisfies
                                                                          一交錯級數審斂法
                                                                                        satisfies the conditions
                                                                                                                                                    then the series is convergent.
                                           斂散性判斷法 (沒有變數的級數)
                                                                                                                                                                                                                                                                          |R_n| = |s-s_n| \le b_{n+1}
                                   Test for Divergence(起手式)
                                                                                                                                                                            條件收斂級數可以透過重排改變和!
                                                                            (拿到先加絕對值)
                                                                                                                                                 (此時稱原級數條件收斂)
                         If \lim a_n doesn't exist or if \lim a_n \neq 0,
                         then the series is divergent.
                                                                                   If \lim_{n \to \infty} |\frac{a_{n+1}}{a_m}| = L < 1, then the series is absolutely convergent.
                                                                          -Ratio Test If \lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = L > 1 or \lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = \infty, then the series is divergent.
                                                                                   |\inf_{n	o\infty}|rac{a_{n+1}}{a_n}|=1, 	ext{ the Ratio Test is inconclusive.}
                                                                                                                                                                                       如果 Ratio Test 結果等於 1
                                                                                                                                                                                       那就不需要試Root Test
                                                                                                                                                                                       結果會等於1! (反之亦然)
                                                                                   If \lim_{n\to\infty}\sqrt[n]{|a_n|}=L<1, then the series is absolutely convergent.
                                                                         Root Test \lim_{n	o\infty}\sqrt[n]{|a_n|}=L>1 	ext{ or }\lim_{n	o\infty}\sqrt[n]{|a_n|}=\infty, 	ext{ then the series is divergent.}
                                                                                   \lim_{n	o\infty}\sqrt[n]{|a_n|}=1, 	ext{ the Root Test is inconclusive.}
                                                                                                             Ratio Test(最常用)
Root Test
                                                                                                                                                收斂區間端點需另外判斷
                                                                                                一收斂區間與半徑-
                                                                                                                         For a power series, there are only three possibilities:
                                                                                                                         1. The series converges only when x = a.
                                                                                                                         ^{\sharp} 2. The series converges for all x.
                                                                                                                         3. There is a number R > 0 such that the series converges if |x - a| < R and diverges if |x - a| > R.
                                                                                               –Geometric Series rac{1}{1-x}=1+x+x^2+\cdots=\sum_{}^\infty x^n |x|<1
                                                                                                                  If the power series \sum c_n(x-a)^n has a radius of convergence R>0,
                                              P.S.的斂散性會與x代入的值有關
                                                                                                                  then the function f defined by f(x) = c_0 + c_1(x-a) + \cdots = \sum c_n(x-a)^n
                                                                                                                                                                                                                       將函數轉為級數是為了:
                                               使P.S.收斂的x值會形成x的收斂區間
                                                                                                                                                                                                                        1.積分沒有 Elementary Antiderivatives 的函數
                                              而收斂區間的中點必為 P.S. 的中心 c。
                                                                                                                                                                                                                        2. Approximating Functions by Polynomials.
                                                                                                                   is differentiable on the interval (a - R, a + R) and
                                                                                             Power Series 微分和積分定理 1.f'(x)=c_1+2c_2(x-a)+\cdots=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}
                                                                                                                                                                                                                        透過幾何級數和定理我們可以得到其他函數的級數
                                                                                                                 2. \int f(x) \, dx = C + co(x-a) + c_1 rac{(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty} c_n rac{(x-a)^{n+1}}{n+1}
                                           冪級數 (Power Series,簡寫 P.S.): \sum_{n=0}^{\infty} a_n (x-c)^n 带有變數 \mathbf{x},次數無窮大的多項式。 \sum_{n=0}^{\infty} a_n (x-c)^n .
                                                                                                                   The radii of convergence of the P.S. in 1 and 2 are both R.
                                                                                                                                       If f has a power series representation at a, that is, if
                                                                                                                                                  f(x) = \sum_n c_n (x-a)^n \quad |x-a| < R
                                                                                                                                       then its coefficients are given by the formula
                                                                                                                                                                                                                                                                                            第二個定理則幫助我們確定函數可不可以用幂級數表示。
                                                                                                                                                                                                                                                                                           第三個定理則是幫助我們去證明定理二中的 \lim R_n = 0
                                                                                                                                                                                                                                                                                           △ 在使用定理二、三時常用到以下極限(請背起來):
                                                                                                                                                                                                                                                                                                     \lim_{n	o\infty}rac{w}{n!}=0 \qquad 	ext{ for every } x\in\mathbb{R}
                                                                                                                                      \overline{\operatorname{If} f(x)} = \overline{T_n(x)} + \overline{R_n(x)}, 	ext{ where } \overline{T_n} 	ext{ is the $n$th-degree Taylor polynomial of $f$ at $a$,}
                                                                                                                               R_n(x) = 0 	ext{ for } |x-a| < R, 	ext{ then } f 	ext{ is equal to the sum of its Taylor series on the interval } |x-a| < R.
                                                                                                                                                    If |f^{(n+1)}(x)| \leq M for |x-a| \leq d then the remainder R_n(x) of the Taylor series satisfies the inequality
                                                                                                                              🗕 🕞 Taylor's Inequality
                                                                                                                                                                                      |R_n(x)| \leq rac{M}{(n+1)!} |x-a|^{n+1} \qquad 	ext{for } |x-a| \leq d
                                                                                                Taylor and Maclaurin Series -
                                                                                                                                                                      \int rac{1}{1-x} = 1+x+x^2+\cdots = \sum_{n=0}^\infty x^n \quad |x| < 1
                                                                                        \operatorname{Taylor}: \sum_{n=0}^{\infty} rac{f^{(n)}(c)}{n!} (x-c)^n
                                                                                                                                                                     -e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} for all x
                                                                                     lackbox{Maclaurin}: \sum_{n=0}^{\infty} rac{f^{(n)}(0)}{n!} x^n
                                                                                                                                                                     -\sin x = x - rac{x^3}{3!} + rac{x^5}{5!} - rac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n rac{x^{2n+1}}{(2n+1)!} \quad 	ext{ for all } x
                                                                                          ⚠Maclaurin Series 是泰勒級數的特例
                                                                                                                              Taylor's's Series of Important Functions -\cos x=1-rac{x^2}{2!}+rac{x^4}{4!}-rac{x^6}{6!}+\cdots=\sum_{n=0}^\infty (-1)^nrac{x^{2n}}{(2n)!} for all x
                                                                                                                                                                                                                                                                        泰勒級數可以拿來算特殊極限!此外
                                                                                                                                                                                                                                                                         我們也可以透過操作這些已知的級數來
                                                                                                                                                                                                                                                                        得到新的泰勒級數
                                                                                                                                                                      -	an^{-1}x = x - rac{x^3}{3} + rac{x^5}{5} - rac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n rac{x^{2n+1}}{(2n+1)} \quad |x| < 1
                                                                                                                                                                      -\ln(1+x) = x - rac{x^2}{2} + rac{x^3}{3} - rac{x^4}{4} + \dots = \sum_{n=0}^{\infty} (-1)^{n-1} rac{x^n}{n} \quad |x| < 1
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 $(1+x)^k=1+kx+rac{k(k-1)}{2!}x^2+\cdots=\sum_{n=0}^\infty inom{k}{n}x^n \quad |x|<1 ext{ and } k\in\mathbb{R}$