參數方程式

Parametric Equations

Parametric equations are equation such that x and y are both functions of a third variable t (call a parameter):

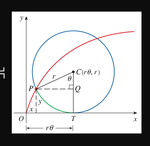
$$x = f(t),\ y = g(t),\ t \in [a, b].$$

Parametric Curve Each t determines a point (x, y). As t varies, $(x, y) = (f(t), g(t))$ varies and traces out a curve, which we call a parametric curve.

Cycloid 擺線

定義 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.

圖片



參數化過程

We choose the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin) as parameter. From the figure, we can know that $|OT| = \text{arc } PT = r\theta$. Therefore the center of the circle is $C(r\theta, r)$. Let coordinates of P be (x, y).

說明 Then from the figure we see that $\begin{cases} x = |OT| - |PQ| = r\theta - r\sin\theta \\ y = |TC| - |QC| = r - r\cos\theta \end{cases}$. Therefore parametric equations of the cycloid are $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$ $\theta \in \mathbb{R}$.

Calculus with Parametric Curves

Tangents

Suppose f and g are differentiable functions and we want to find the slope of a tangent line at a point on the parametric curve $x = f(t),\ y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

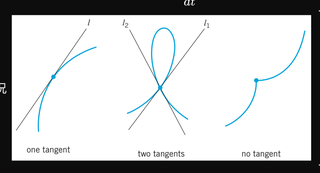
為了瞭解這個參數方程曲線的凹性，我們有時也需要計算二階導數 $\frac{d^2y}{dx^2}$：

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

切線之斜率計算

更直觀來看的話，可以把 $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ 想成是切向量的 y 分量除以 x 分量！

平面曲線的切線可能情況



判斷是否有切線的方法

(1)若 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 不同時為 0，則必存在一條切線！若 $\begin{cases} \frac{dy}{dt} = 0 \\ \frac{dx}{dt} \neq 0 \end{cases}$，則 C 有水平切線。

若 $\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} \neq 0 \end{cases}$，則 C 有垂直切線。

(2)若 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 同時為 0，什麼情形都有可能發生，必須另外找方法判斷。

兩者皆為零之說明範例

Verify that the following examples satisfy the equation $f'(0) = g'(0) = 0$ and that the given description holds at the point where $t = 0$.

1. $f(t) = t^3,\ g(t) = t^2$; cusp.
2. $f(t) = t^3,\ g(t) = t^5$; horizontal tangent.
3. $f(t) = t^5,\ g(t) = t^3$; vertical tangent.
4. $f(t) = t^3 - 1,\ g(t) = 2t^3$; tangent with slope 2.
5. $f(t) = t^2,\ g(t) = t^2 + 1$; no tangent line.

Area Under Parametric Curves

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, if $F(x) \geq 0$.

If the curve can be traced out by the parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases},\ \alpha \leq t \leq \beta,$ then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

計算方式

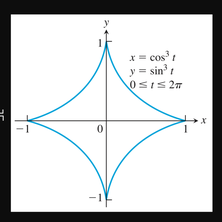
Let $y = F(x)$ be corresponding Cartesian equation. Suppose $a \leq x \leq b$ corresponds to $\alpha \leq t \leq \beta$.

(1) If $t = \alpha$ derives leftmost endpoint, then $A = \int_\alpha^\beta g(t)f'(t)\,dt$.

(2) If $t = \beta$ derives leftmost endpoint, then $A = \int_\beta^\alpha g(t)f'(t)\,dt$.

範例 Find the area enclosed by the astroid.

圖片



做法

$$\begin{aligned} A &= 4\int_0^1 y\,dx && \text{4 times area under } y \text{ from } x = 0 \text{ to } x = 1 \\ &= 4\int_0^{\frac{\pi}{2}} (\sin^3 t)(3\cos^2 t \sin t)\,dt && \text{Substitution for } y \text{ and } dx \\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt && \sin^4 t = \left(\frac{1 - \cos 2t}{2}\right)^2 \\ &= \frac{3\pi}{8} \end{aligned}$$

Cartesian Coordinates

If curve C is given by $x = f(t),\ y = g(t),\ \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once, then length of C is $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

計算方式

範例 Calculate the circumference of the unit circle.

作法一

Parametrizing the unit circle by setting $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases},\ t \in [0, 2\pi]$, we have $\begin{cases} x'(t) = -\sin t \\ y'(t) = \cos t \end{cases}$

and the arc length formula gives $L(C) = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$

作法二

Parametrizing the unit circle by setting $\begin{cases} x(t) = \cos 2t \\ y(t) = \sin 2t \end{cases},\ t \in [0, 2\pi],$ we have $\begin{cases} x'(t) = -2\sin 2t \\ y'(t) = 2\cos 2t \end{cases}$

and the arc length formula gives $L(C) = \int_0^{2\pi} \sqrt{4\sin^2 2t + 4\cos^2 2t}\,dt = \int_0^{2\pi} 2\,dt = 4\pi.$

很明顯，第一個作法算出來的周長才是對的。為什麼第二個作法會錯呢？
因為這個公式算出來的會是實際經過的長度，第一個作法只有畫一圈，但是第二個作法畫了兩圈，因此得出兩倍周長！我們之前有說過參數化曲線的方式不只一種，你可以用其他參數方程表示同一條曲線，但若要用參數方程算曲線長度，請記得這個參數方程只能經過曲線上的每個點一次！

Arc Length

Suppose that C is the graph of a continuously differentiable function $y = (x),\ x \in [a, b]$.

和定積分弧長公式比較

We can parametrize C by setting $\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases}$ $t \in [a, b]$. Since $x'(t) = 1$ and $y'(t) = f'(t)$,

gives $L(C) = \int_a^b \sqrt{1 + [f'(t)]^2}\,dt$. Replacing t by x, we can write: $L(C) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$.

這個式子跟我們前面用積分算 Arc Length 的式子是一樣的，這兩個方法的差異在於前面教的公式只能用在 C is the graph of a continuously differentiable function $y = (x)$ 的情況，但後者則沒有這個限制！

Arc Length Function

For a parametric curve C given by $x = f(t),\ y = g(t)$, where f' and g' are continuous, we let $s(t)$ be the arc length along C from an initial point $(f(\alpha), g(\alpha))$ to a point $(f(t), g(t))$ on C. By formula, the arc length function s for parametric curves is $s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du$

Surface Area

If curve is given by $x = f(t), y = g(t), a \leq t \leq b$ and is rotated about $x-$axis, where f', g' are continuous and $g(t) \geq 0$, then resulting surface area S is

$$S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\,.$$

If curve is given by $x = f(t), y = g(t), a \leq t \leq b$ and is rotated about $y-$axis, where f', g' are continuous and $f(t) \geq 0$, then resulting surface area S is

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\,.$$

和定積分公式比較

我們以繞 x 軸為例。Suppose that C is the graph of a continuously differentiable nonnegative function $y = f(x),\ x \in [a, b]$.

We can parametrize C by setting $\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases}$ $t \in [a, b]$. Since $x'(t) = 1$ and $y'(t) = f'(t)$, the area of surface generated by revolving C about the x axis is given by the formula $A = \int_a^b 2\pi y\sqrt{1 + [f'(t)]^2}\,dt$. Replacing t by x, we can write: $A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$.

這個式子跟我們前面用積分算 Surface Area 的式子是一樣的，這兩個方法的差異在於前面教的公式只能用在 C is the graph of a continuously differentiable nonnegative function $y = (x)$ 的情況，但後者則沒有這個限制！

Polar Coordinates

Tangents

If a curve C is given by $r = f(\theta)$, we can write parametric equations for a polar curve $r = f(\theta)$ as follows

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta \text{（轉換成直角坐標下的參數方程式）}$$

根據參數方程式的切線斜率公式，我們可以得到 $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

判斷是否有切線的方法

(1)若 $\frac{dx}{d\theta}$ 和 $\frac{dy}{d\theta}$ 不同時為 0，則必存在一條切線！若 $\begin{cases} \frac{dy}{d\theta} = 0 \\ \frac{dx}{d\theta} \neq 0 \end{cases}$，則 C 有水平切線。

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(2)若 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 同時為 0，什麼情形都有可能發生，必須另外找方法判斷。

Area

Let R denote the region bounded by $r = f(\theta),\ \theta = a$ and $\theta = b$, where $f(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. Then the area A of R is

$$A = \int_a^b \frac{1}{2}(f(\theta))^2\,d\theta = \int_a^b \frac{1}{2}r^2\,d\theta$$

Arc Length

If a curve C is given by $r = f(\theta)$, we can write parametric equations for a polar curve $r = f(\theta)$ as follows

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta \text{（轉換成直角坐標下的參數方程式）}$$

因為 $\begin{cases} \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{cases} \xrightarrow{\text{Using } \cos^2\theta + \sin^2\theta = 1} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

所以，我們可以把弧長公式 $L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$ 寫成 $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\,.$

總而言之

A curve with polar equation $r = f(\theta),\ a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\,.$$