單變數純量函數瑕積分

定義

積分區間無限或是積分函數無限的定積分就叫做瑕積分。
接下來就是要介紹如果遇到瑕積分該怎麼做的技巧。

> 我們之前學過，如果定積分的積分範圍是 $[a,b]$，則函數可積的條件至少是「在積分範圍內有界」或是「函數在積分範圍內只有有限多個 jump 型的不連續點」，但瑕積分所考慮的積分區間有可能是 $[a,\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$，函數也可能會在積分範圍內跑到正負無窮大 （不是 jump 型的不連續），所以「瑕積分的積分值不一定存在」。如果瑕積分的積分值存在，我們就稱此瑕積分收斂 (Convergent)，如果它的積分值不存在，我們就稱此瑕積分發散 (Divergent)。請注意，「瑕積分的積分值不存在」指的是計算出來的結果振盪（即值不唯一）或是計算出來的結果跑到正負無窮大。

種類

Infinite Intervals

1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ provided the limit exists.

2. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$ provided the limit exists.

3. If both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \quad \forall\, a \in \mathbb{R}$$

常見錯誤說明

1. 自創寫法：
$$\int_0^\infty x^3\,dx = \frac{1}{4}x^4\Big|_0^\infty = \infty \quad \text{雖然答案對，但寫法不對！}$$
2. 自創定義：
$$\int_{-\infty}^\infty x\,dx = \lim_{t\to\infty}\int_{-t}^t x\,dx = \lim_{t\to\infty}\frac{x^2}{2}\Big|_{-t}^t = \lim_{t\to\infty}\left(\frac{t^2}{2} - \frac{(-t)^2}{2}\right) = 0 \quad \text{寫法不對！}$$
3. 自創性質：
$$f(x) \text{ is an odd function} \Rightarrow \int_{-\infty}^\infty x\,dx = 0 \quad \text{寫法不對！其實} \int_{-\infty}^\infty x\,dx \text{ 發散！}$$

> 總而言之，就是一定要照定義算，不要自己幻想一些奇怪的性質或公式甚至是靠直覺！

Proposition

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx \quad \forall\, a < c$, provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall\, c < b$, provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

Discontinuous Integrands

1. If f is Riemann integrable on $[a,c]$ $\forall\, a < c < b$ and has an infinite discontinuity at b, then $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$ provided the limit exists.

2. If f is Riemann integrable on $[c,b]$ $\forall\, a < c < b$ and has an infinite discontinuity at a, then $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$ provided the limit exists.

3. If f has an infinite discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

常見錯誤說明

1. 看到積分範圍有限就沒有檢查是不是瑕積分：
$$\int_{-1}^3 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{-1}^3 = \frac{-4}{3} \quad \text{寫法不對！其實} \int_{-1}^3 \frac{1}{x^2}\,dx \text{ 發散！}$$
2. 自創寫法：
$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = 2\sqrt{x-2}\Big|_2^5 = 2\sqrt{3} - 2\sqrt{0} = 2\sqrt{3} \quad \text{寫法不對！}$$
請試著算算看，正確答案雖然還是 $2\sqrt{3}$，但這只是剛好，一定要使用定義來計算！

> 從現在開始看到積分就要注意它是不是瑕積分，而不是看到題目就積分。

Proposition

If f is Riemann integrable on $[a,c]$ $\forall\, a < c < b$, and f has an infinite discontinuity at a or b, then $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall a < c < b$, provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

Infinite Intervals & Discontinuous Integrands

例：$\int_{-\infty}^\infty \frac{1}{x^2}\,dx$

> 把它想辦法拆解成前面兩種的其中一種，就可以算了！

Theorem

Direct Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

P.S. Similar result also holds for other kinds of improper integrals.

Limit Comparison Test

Let f and g be positive continuous functions on the interval $[a,\infty)$.
If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$ for some $0 < L < \infty$, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^\infty \mathrm{g(x)\,dx}$ converges.

P.S. Similar result also holds for other kinds of improper integrals.

> 有時候沒辦法找到瑕積分的值，但知道它是收斂還是發散很重要！

常見瑕積分

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{發散} & p \leq 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{發散} & p \geq 1 \\ \frac{1}{1-p} & p < 1 \end{cases}$$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \begin{cases} \text{發散} & p \leq 1 \\ \frac{1}{(p-1)^2} & p > 1 \end{cases}$$

$$\int_0^1 \frac{\ln x}{x^p}\,dx = \begin{cases} \text{發散} & p \geq 1 \\ \frac{-1}{(1-p)^2} & p < 1 \end{cases}$$

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \begin{cases} \text{發散} & p \leq 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$