單變數純量函數積分（概念篇）

Definite Integral

先備知識：Sigma

Proposition

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Sums of Powers

$$\sum_{i=1}^{n} c = cn \quad \text{if } c \text{ is a constant}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

前言

Let $f\colon [a,b] \to \mathbb{R}$ be a non-negative continuous function, and R be the region enclosed by the graph of the function f, the x-axis and straight lines $x=a$ and $x=b$. We consider computing $A(R)$, the area of R.

計算曲線下的面積—第一步（考慮非負連續函數）

Partition $[a,b]$ into n sub-intervals with equal length, and let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$.

By the Extreme Value Theorem, for each $1 \le i \le n$, f attains its maximum and minimum on $[x_{i-1}, x_i]$; thus for $1 \le i \le n$, there exist $M_i, m_i \in [x_{i-1}, x_i]$ such that $\begin{cases} f(M_i) = \text{the maximum of } f \text{ on } [x_{i-1}, x_i] \\ f(m_i) = \text{the minimum of } f \text{ on } [x_{i-1}, x_i] \end{cases}$

計算曲線下的面積—第二步（考慮非負連續函數）

$\begin{cases} S(n) \equiv \sum_{i=1}^{n} f(M_i)\Delta x \text{ is called the upper sum of } f \\ s(n) \equiv \sum_{i=1}^{n} f(m_i)\Delta x \text{ is called the lower sum of } f \end{cases}$ for the partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$,

By the definition of the upper sum and lower sum, we find that for each $n \in \mathbb{N}$, $\sum_{i=1}^{n} f(m_i)\Delta x \le A(R) \le \sum_{i=1}^{n} f(M_i)\Delta x$.

If the limits of the both sides exist and are identical as Δx approaches 0 (which is the same as n approaches infinity), by the Squeeze Theorem we conclude that $A(R)$ is the same as the limit.

Riemann Integrals

Partition

A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is said to be a partition of the closed interval $[a,b]$ if $a = x_0 < x_1 < \cdots < x_n = b$.
Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n = b\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ is the number $\max\{x_i - x_{i-1} \mid 1 \le i \le n\}$; that is, $\|\mathcal{P}\| \equiv \max\{x_i - x_{i-1} \mid 1 \le i \le n\}$. A partition is called regular if $x_i - x_{i-1} = \|\mathcal{P}\|$ for all $1 \le i \le n$.

Riemann Sums

Let $f\colon [a,b] \to \mathbb{R}$ be a function. A Riemann sum of f for the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ is a sum which takes the form $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$, where the set $S = \{c_0, c_1, \cdots, c_{n-1}\}$ satisfies that $x_{i-1} \le c_i \le x_i$ for each $1 \le i \le n$.

$c_i \in [x_{i-1}, x_i]$是可以任選的，且其被稱為 Sample Points。

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a,b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A is unique and is called the Riemann integral of f on $[a,b]$ and is denoted by $\displaystyle\int_{[a,b]} f(x)\,dx$.

Riemann Integral 存在判準一：

連續必可積

If f is continuous on $[a,b]$, or f has only a finite number of jump discontinuous, then f is Riemann integrable on $[a,b]$.

有界必有積

Let $f\colon [a,b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a,b]$, then f is bounded on $[a,b]$; that is, there exists $M > 0$ such that $|f(x)| \le M$ whenever $x \in [a,b]$.

事實上，“f is continuous on $[a,b]$，意即”f 在 $[a,b]$ 上是連續函數，所以下面的所要表達方式有：

(1) 函數 f 是在閉區間中有界，即 f 不一定可積，例如：for all $x \in [0,1]$,

$$f(x) = \begin{cases} 0 & , \text{if } x \in \mathbb{Q}, \\ 1 & , \text{if } x \notin \mathbb{Q}. \end{cases}$$

這個的 f 不可積。

(2) 函數 f 在閉區間中只有有限個不連續（這裡的不連續專指 jump discontinuous），則 f 仍可積，例如：

$$f(x) = \begin{cases} 1 & , \text{if } 0 \le x \le 2, \\ 2 & , \text{if } 2 < x \le 3, \end{cases}$$

這個的 f 是可積的。

定義

For conventional reason, the Riemann integral of f over the interval with left-end point a and right-end point b is written as $\displaystyle\int_a^b f(x)\,dx$, and is called the definite integral of f from a to b. We also note that here in the representation of the integral, x is a dummy variable; that is, we can use any symbol to denote the independent variable; thus $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$.

$\displaystyle\int$: Integral Sign $\quad f(x)$: Integrand $\quad a$: Lower Limit $\quad b$: Upper Limit

範例

曲線下的面積（連續非負函數）

Let $f\colon [a,b] \to \mathbb{R}$ be a non-negative and continuous function. The area of the region enclosed by the graph of f, the axis, and the vertical lines $x=a$ and $x=b$ is $\displaystyle\int_a^b f(x)\,dx$.

非負連續函數與有正有負的連續函數在定積分計算下的差異

(1) $\displaystyle\int_a^b f(x)\,d$ can be interpreted as the area if $f(x) \ge 0$ for all $x \in [a,b]$.

(2) $\displaystyle\int_a^b f(x)\,d$ can be interpreted as the net area if $f(x) \in \mathbb{R}$（f 有正有負） for all $x \in [a,b]$.

EXAMPLE Evaluate $\int_0^4 x^{\frac{1}{2}}\,dx$.

SOLUTION We use Theorem 6.6.7 with $f(x) = x^{\frac{1}{2}}$, $a_0 = 0$, $b = 4$, and $q_{i-1} = \frac{4}{n}i^2$, $i = 1, 2, \ldots, n$.

Then the endpoints of the subintervals are $x_0 = 0$, $x_1 = 4$, $x_2 = 16$, $x_3 = 36$, $x_4 = 64$, $x_5 = 100$, $x_6 = 144$, $x_7 = 196$, $x_8 = 256$, and in general

$$x_i = 4 + \left(\frac{4}{n}\right)i^2 \quad i = 1, 2, \ldots, n$$

Thus

$$\begin{aligned}
\int_0^4 x^{\frac{1}{2}}\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}i^2\right)^{\frac{1}{2}} \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)^{\frac{1}{2}} \left(i^2\right)^{\frac{1}{2}} \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)^{\frac{1}{2}} i^{\left(2\cdot\frac{1}{2}\right)} \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\right)^{\frac{1}{2}} \cdot \frac{4}{n} \cdot \sum_{i=1}^{n} i \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\right)^{\frac{1}{2}} \left[\frac{n(n+1)}{2}\right] \cdot \frac{4}{n} = \lim_{n\to\infty} 2\left(\frac{4}{n}\right)^{\frac{1}{2}} \left[\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\right)^{\frac{1}{2}} \left(n^2 + n\right)^{\frac{1}{2}} = \lim_{n\to\infty} \left(\frac{4}{n}\right)^{\frac{1}{2}} n^{\left(2\cdot\frac{1}{2}\right)} \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} \\
&= 2^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} \cdot 1^{\frac{1}{2}} = 4\sqrt{2}
\end{aligned}$$

Proposition

Integral Rules Suppose that f is continuous on $[a,b]$.

$$\int_a^b c\,dx = c(b-a) \quad \text{where } c \text{ is a constant}$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{where } c \text{ is an constant}$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx \quad (a < c < b)$$

Corollary

Suppose f is continuous on $[-a,a]$.

1. If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
2. If f is even $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Comparison Properties

If $f(x) \ge 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx \ge 0$.

If $f(x) \ge g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

If $m \le f(x) \le M$ for all $x \in [a,b]$, then $m(b-a) \le \displaystyle\int_a^b f(x)\,dx \le M(b-a)$

Definite Integral as a Punction of Upper Limit

Prerequirement

Until now we have integrated only from left to right: from a number a to a number b greater than a. We integrate in the other direction by defining $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

The integral from any number to itself is defined to be zero: $\displaystyle\int_c^c f(x)\,dx = 0$ if f is defined at $x = c$

With this additional conventions, the additivity condition $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ holds for all choices of a, b, c from an interval on which f is continuous, no matter what the order of a, b, c happens to be.

Definition

If f is continuous on $[a,b]$ and c is any number in $[a,b]$, then for each x in $[a,b]$, the integral is a number, and consequently we can define a function F on $[a,b]$ by setting $F(x) = \displaystyle\int_c^x f(t)\,dt$.

Theorem

Let f be continuous on $[a,b]$ and let c be any number in $[a,b]$. The function F defined on $[a,b]$ by setting $F(x) = \displaystyle\int_c^x f(t)\,dt$ is continuous on $[a,b]$, differentiable on (a,b), and has derivative $F'(x) = f(x)$ for all x in (a,b).

The Fundamental Theorem of Calculus 微積分基本定理

定理內容

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is an antiderivative of f, that is, a function F such that $F' = f$.

Antiderivative A function F is called an antiderivative on an interval I if $F'(x) = f(x)$ for all x in I.

概念補充 Suppose that g, h is a differentiable function of x with values in $[a,b]$.

1. $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ and then $\dfrac{d}{dx}\displaystyle\int_x^a f(t)\,dt = -\dfrac{d}{dx}\int_a^x f(t) = -f(x)$
2. $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$
3. $\dfrac{d}{dx}\displaystyle\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$

Mean Value Theorem for Integral

First

If f is continuous on $[a,b]$, then there is at least one number $c \in (a,b)$ for which $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$.

This number $f(c)$ is called the average value of f on $[a,b]$.

Second

If f and g are continuous on $[a,b]$, and g is nonnegative, then there is a number $c \in (a,b)$ for which $\displaystyle\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

This number $f(c)$ is called the g-weighted average of f on $[a,b]$.

Indefinite Intergral

Definition

We begin with a continuous function f. If F is an antiderivative for f on [a,b], then $\displaystyle\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$.
If C is a constant, then $\left[F(x) + C\right]_a^b = \left[F(b) + C\right] - \left[F(a) + C\right] = F(b) - F(a) = \left[F(x)\right]_a^b$.
Thus we can write $\displaystyle\int_a^b f(x)\,dx = \left[F(x) + C\right]_a^b$. If we have no particular interest in the interval $[a,b]$ but wish instead to emphasize that F is an antiderivative for f, which on open intervals simply means $F' = f$, then we omit the a and b and simply write $\displaystyle\int f(x)\,dx = F(x) + C$. Antiderivative expressed in this manner are called indefinite integrals.

不定積分表

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int a\,dx = ax + C$	
$\int x^r\,dx = \dfrac{x^{r+1}}{r+1} + C \quad (\text{if } r \ne -1)$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \dfrac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}\,dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
$\int \cosh x\,dx = \sinh x + C$	$\int \sinh x\,dx = \cosh x + C$

不定積分法

Substitution 變數置換

不定積分版

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx \xrightarrow{du = g'(x)dx} \int f(u)\,du \xrightarrow{F' = f} F(u) + C$$

定積分版

If $g'(x)$ is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx \xrightarrow{du = g'(x)dx} \int_{g(a)}^{g(b)} f(u)\,du \xrightarrow{F' = f} \left[F(u)\right]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

變數置換是從 Chain Rule 來的，
說明如下：

$$\int f(g(x))g'(x)\,dx \xrightarrow{F' = f} \int F'(g(x))g'(x)\,dx \xrightarrow{\text{by the Chain Rule}} \int \frac{d}{dx}\left[F(g(x))\right]dx \xrightarrow{\text{by 微積分基本定理}} F(g(x)) + C.$$

⚠ 其實左邊的寫法只是為了方便計算，並不嚴謹，因為積分符號的 dx 並不是 Differential，正確的寫法應該是從 Chain Rule 去看。
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx \xrightarrow{F' = f} \int F'(g(x))g'(x)\,dx \xrightarrow{\text{by the Chain Rule}} \int \frac{d}{dx}\left[F(g(x))\right]dx \xrightarrow{\text{by 微積分基本定理}} F(g(x)) + C.$$

範例

Find $\displaystyle\int 2x\sqrt{1+x^2}$.

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx$$

不定積分

$$\begin{aligned}
&= \int \sqrt{u}\,du \qquad \text{Let } u = 1 + x^2 \text{ then } du = 2xdx \\
&= \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C
\end{aligned}$$

定積分

Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\,dx$ $\begin{cases} \text{法一是先代入新變數，算完之後再變回舊變數。} \\ \text{法二則是先代入新變數後，把積分範圍換成新變數的積分範圍。} \end{cases}$

法一：$\displaystyle\int_0^4 \sqrt{2x+1} = \int \sqrt{2x+1}\Big|_{x=0}^{x=4}$ Let $u = 2x+1$. Then $du = 2dx$, so $dx = \dfrac{1}{2}du$.

$$\begin{aligned}
&= \int \sqrt{u}\cdot\frac{1}{2}\,du\Big|_{x=0}^{x=4} = \frac{1}{2}\int u^{\frac{1}{2}}\,du\Big|_{x=0}^{x=4} \\
&= \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C\Big|_{x=0}^{x=4} = \frac{1}{3}u^{\frac{3}{2}} + C\Big|_{x=0}^{x=4} \qquad u = 2x+1 \\
&= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C\Big|_{x=0}^{x=4} = \frac{26}{3}
\end{aligned}$$

法二：Let $u = 2x+1$. Then $du = 2dx$, so $dx = \dfrac{1}{2}du$. When $x = 0, u = 2(0)+1 = 1$ When $x = 4, u = 2(4)+1 = 9$

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=9} = \frac{26}{3}$$

Integration by Parts 分部積分

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

不定積分版 If we let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.
We can write $\displaystyle\int u\,dv = uv - \int v\,du$.

分部積分是從Product Rule for Differentiation 來的，
說明如下：

$$\begin{aligned}
&\frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x) \\
&\Rightarrow f(x)g(x) = \int \{f(x)g'(x) + g(x)f'(x)\}\,dx \\
&\Rightarrow f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx
\end{aligned}$$

定積分版

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

一些積分技巧

透過微積分基本定理和一些微分法則，我們可以有這些積分方法！