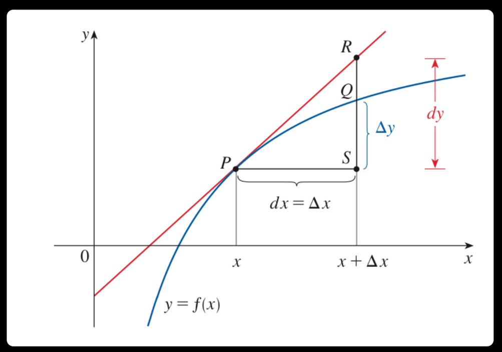
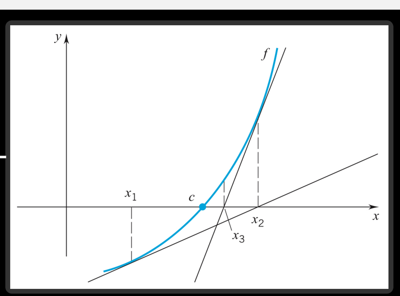
單變數純量函數導數（應用篇）

算近似值

- **Linear Approximation**
  - 之前有提到可微則存在切線，我們可以用切線作線性近似，說明如下。
  - If f is differentiable at $x=a$,
    then the approximating function $L(x)=f(a)+f'(a)(x-a)$ is the linearization of f at a.
    The approximation $f(x)\approx L(x)$ is called the linear approximation of f at a.
  - Differential: The idea behind linear approximations are sometimes formulated in the terminology and notation of differentials.
    If $y=f(x)$, where f is a differentiable, then the differential dx is an independent variable;
    that is, dx can be given the value of any real number.
    The differential dy is then defined in terms of dx by the equation $dy=f'(x)dx$
    so dy is an dependent variable; it depends on the values of x and dx.



- **Newton's Method**: The method of locating a root of an equation $f(x)=0$ is call the Newton-Raphson method.
  - 概念: c is the solution of the equation $f(x)=0$. We can approximate c as follows:
    Start at x_1. The tangent line at $(x_1, f(x_1))$ intersects the x-axis at a point x_2
    which is closer to c than x_1. The tangent line at $(x_2, f(x_2))$ intersects the x-axis
    at point x_3 which in turn is closer to c than x_2. In this manner, we obtain numbers
    $x_1,\dots,x_n,x_{n+1}$, which more and more closely approximate c.



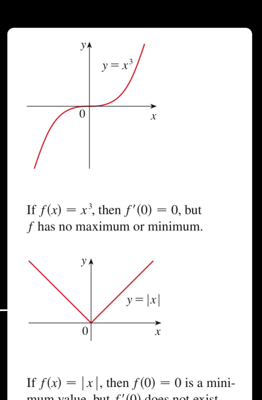
  - 作法: Suppose that f is twice differentiable.
    1. Make an initial estimate $x_1\in(a,b)$ that is close to c. ($f(x)f''(x)>0$ for all x between c and x_1.)
    2. Determine a new approximation using the iterative relation: $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$.
    3. When $|x_n-x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation.

算不定型極限

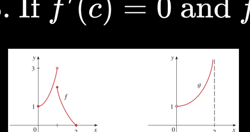
- **Indeterminate Forms（不定型）**
  - Consider $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$：
  - 1.若$x\to a$時，有$f(x)\to 0$及$g(x)\to 0$，則我們稱該極限為「$\dfrac{0}{0}$不定型」.
  - 2.若$x\to a$時，有$f(x)\to\infty$及$g(x)\to\infty$，則我們稱該極限為「$\dfrac{\infty}{\infty}$不定型」.(∞可改為$-\infty$)
  - 上述所提及的$x\to a$可改為$x\to a^+, x\to a^-, x\to\infty$或$x\to-\infty$
- **L'Hospital's Rule（算不定型極限定理）**
  - Suppose f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contains a (except possibly at a).
    若$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$屬於$\dfrac{0}{0}$或$\dfrac{\infty}{\infty}$不定型，則$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ if the limit on the right side exists (or is ∞ or $-\infty$).
- **其他不定型**
  - **Indeterminate Products**: Consider $\lim\limits_{x\to a}f(x)g(x)$：
    若$x\to a$時，有$f(x)\to 0$及$g(x)\to\infty$，則我們稱該極限為「$0\cdot\infty$不定型」.(∞可改為$-\infty$)
    - 這種不定型有兩種計算方式，哪個好用就用哪個！
      1.因為$\lim\limits_{x\to a}f(x)g(x)=\lim\limits_{x\to a}\dfrac{f(x)}{\frac{1}{g(x)}}$，所以$0\cdot\infty$不定型可化為$\dfrac{0}{0}$不定型。
      2.因為$\lim\limits_{x\to a}f(x)g(x)=\lim\limits_{x\to a}\dfrac{g(x)}{\frac{1}{f(x)}}$，所以$0\cdot\infty$不定型可化為$\dfrac{\infty}{\infty}$不定型。
  - **Indeterminate Differences**: Consider $\lim\limits_{x\to a}[f(x)-g(x)]$：
    若$x\to a$時，有$f(x)\to\infty$及$g(x)\to\infty$，則我們稱該極限為「$\infty-\infty$不定型」.
    - 此種不定型的計算重點在於把減改成除，再用$\dfrac{0}{0}$或$\dfrac{\infty}{\infty}$不定型來算。
  - **Indeterminate Powers**: Consider $\lim\limits_{x\to a}[f(x)]^{g(x)}$，其中$f(x)>0$：
    (1)若$x\to a$時，有$f(x)\to 0$及$g(x)\to 0$，則我們稱該極限為「0^0不定型」。
    (2)若$x\to a$時，有$f(x)\to\infty$及$g(x)\to 0$，則我們稱該極限為「∞^0不定型」。
    (3)若$x\to a$時，有$f(x)\to 1$及$g(x)\to\infty$，則我們稱該極限為「1^∞不定型」。
    - 此種不定型的計算重點在於「令$y=f(x)^{g(x)}$」，此時$\lim\limits_{x\to a}\ln y=\lim\limits_{x\to a}g(x)\ln f(x)$為$0\cdot\infty$不定型，然後因為$\lim\limits_{x\to a}f(x)^{g(x)}=\lim\limits_{x\to a}y=\lim\limits_{x\to a}e^{\ln y}=e^{\lim_{x\to a}\ln y}$。因此只要能算出$\lim\limits_{x\to a}\ln y$，就能得到$\lim\limits_{x\to a}f(x)^{g(x)}$的答案。

找極值

- **極值介紹**
  - 絕對極值: Let c be a number in the domain D of a function f. Then $f(c)$ is the
    · (Absolute) Maximum value of f on D if $f(c)\geq f(x)$ for all x in D.
    · (Absolute) Minimum value of f on D if $f(c)\leq f(x)$ for all x in D.
  - 局部極值: The number $f(c)$ is a
    · Local Maximum value of f if $f(c)\geq f(x)$ when x is near c.
    · Local Minimum value of f if $f(c)\leq f(x)$ when x is near c.
- **相對極值**
  - **Fermat's Theorem**: If f has a local maximum or minimum at c and $f'(c)$ exists, then f'(c)=0.
    - ⚠️謹記：若 P 則 Q；非 Q 則非 P！
      1.就算f'(c)=0，也有可能沒有局部極值！
      2.就算f'(c)不存在，也有可能有局部極值！



  - 綜合上述，我們發現極值不是在$f'(c)=0$就是在$f'(c)$不存在的點上，因此，可以把 Fermat's Theorem 改成如下：
    If f has a local maximum or minimum at c,
    then c is a critical number of f.
    - Critical Number: A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ doesn't exists.
  - **Increasing/Decreasing**
    - 定義: Let f be defined on an interval I.
      1. f is increasing on I if for all $x_1,x_2\in I, x_1<x_2$ implies $f(x_1)<f(x_2)$.
      2. f is decreasing on I if for all $x_1,x_2\in I, x_1<x_2$ implies $f(x_1)>f(x_2)$.
      - A function that is increasing or decreasing on an interval is said to be monotonic on the interval.
    - 判別方式: Let f be continuous on a closed interval $[a,b]$ and f be differentiable on an open interval (a,b).
      1.If $f'(x)>0$ for all $x\in(a,b)$, then f is increasing on $[a,b]$.
      2.If $f'(x)<0$ for all $x\in(a,b)$, then f is decreasing on $[a,b]$.
      3.If $f'(x)=0$ for all $x\in(a,b)$, then f is constant on $[a,b]$.
  - **The First Derivative Test（找相對極值方法 1）**: Suppose that c is a critical number of a continuous fumction f.
    1.If f' changes from positive to negative at c, then f has a local maximum at c.
    2.If f' changes from negative to positive at c, then f has a local minimum at c.
    3.If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local extrema at c.
  - 前面的定理只告訴我們若該點有相對極值，該點一定是 Critical Number 上。因此我們需要 The First Derivative Test，來告訴我們在 Critical Number 是否有相對極值！
  - **Concave Upward/Downward**
    - 定義: Let f be differentiable on an open interval I.
      1.The graph of f is concave upward on I if f' is increasing on I.
      2.The graph of f is concave downward on I if f' is decreasing on I.
    - 判別方式:
      1. If $f''(x)>0$ on an interval I, then the graph of f is concave upward on I.
      2. If $f''(x)<0$ on an interval I, then the graph of f is concave downward on I.
  - **Inflection Point**: Let f be a differentiable on an open interval containing c.
    The point $(c, f(c))$ is called an inflection point of the graph of f
    if the concavity of f changes from upward to downward or downward to upward at this point.
  - **The Second Derivative Test（找相對極值方法 2）**: Suppose that f'' is continuous near c.
    1. If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
    2. If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.
    3. If $f'(c)=0$ and $f''(c)=0$ or doesn't exist, then the test fails.
    - 通常會用 The First Derivative Test！
- **絕對極值**
  - 「有可能」沒有絕對極值的情況



    - 還記得我們在連續函數提到的 Extreme Value Theorem嗎？這邊提醒大家只要極值定理中的其中一個條件沒有被滿足就「有可能」沒有絕對極值。
      1.左邊的圖沒有連續（只有絕對極小）
      2.右邊的圖沒有在閉區間上（沒有絕對極大也沒有絕對極小）
  - **The Closed Interval Method（找絕對極值方法 1）**: If f is continuous on a closed interval$[a,b]$,
    we can follow below to find the absolute maximum and minimum :
    1.Find critical numbers of f in (a,b).
    2.Evaluate f at each critical points in (a,b).
    3.Evaluate f at the end-points in $[a,b]$.
    4.Compare the values of 2 and 3.（這些值中最大的就是絕對極大值；最小的就是絕對極小值！）
  - **The First Derivative Test for Absolute Extrema（找絕對極值方法 2）**: Suppose that c is a critical number of a continuous function f defined on an interval.
    1.If $f'(x)>0$ for all $x<c$ and $f'(x)<0$ for all $x>c$, then $f(c)$ is the absolute maximum value of f.
    2.If $f'(x)<0$ for all $x<c$ and $f'(x)>0$ for all $x>c$, then $f(c)$ is the absolute minimum value of f.
  - 除了這兩種方法，也可以在找到全部的相對極值後，直接和端點比較就知道最大最小值了！

Curve Skeching

- **步驟**
  - Domain : 確定定義域
  - Intercepts : 找出 x 截距、y 截距
  - Symmetry： 確定函數是否具有對稱性
  - Asymptotes : 判斷是否有水平、垂直、斜漸近線
  - Intervals of Increase or Decrease : 找出函數的遞增遞減區間
  - Local Extrema : 利用 Critical Numbers 與一次導數判別法找出局部極值
  - Concavity and Points of Inflection : 利用反曲點確定函數的凹性
  - Sketch the Curve : 綜合上述畫圖
- **範例**