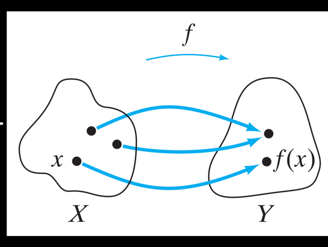
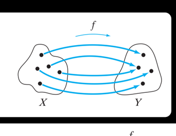
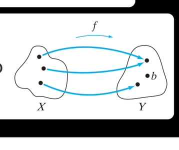
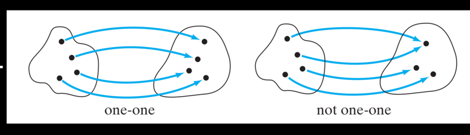
單變數純量函數極限與連續

函數
這邊重新幫大家複習函數的觀念，有助於之後學習。

- **Definition**: Let X and Y be given sets. A function $f: X \to Y$ consists of two sets X and Y together with a "rule" that assigns to each $x \in X$ a special element of Y denoted by $f(x)$. One writes $f(x)$ to denote that x is mapped to the element $f(x)$. X is called the domain（定義域）of f, and Y is called the target or co-domain of f. The range （值域） of f or the image of f, is the subset of Y defined by $f(X) = \{f(x) \mid x \in X\}$.
  P.S. We will frequently use the notation $f : X \to Y$ for a function.

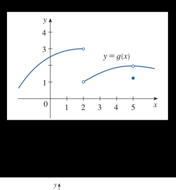
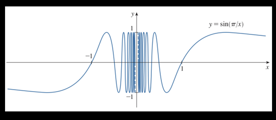


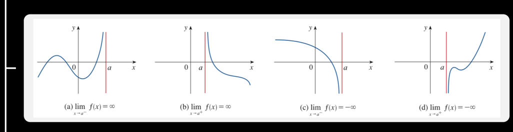
- **Onto (Surjective)**: A funcion $f : X \to Y$ is said to be **onto** if range $f = Y$.
  - Onto 
  - Not Onto 
- **One-One (Injective)**: A function $f : X \to Y$ is called **one-one** if whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.



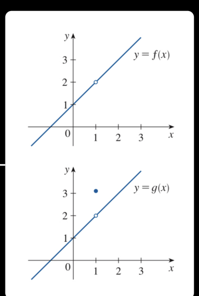
- **Bijection**: A funcion $f : X \to Y$ is called an bijection if it is one-to-one and onto.
- **Composite Function**: Let X, Y, and Z be sets and $f: X \to Y$ and $g: Y \to Z$ be functions. By following f with g, we obtain a function $g \circ f: X \to Z$ called the composite of g and f. Thus $(g \circ f)(x) = g(f(x))\ \forall x \in X$
  - Functional composition is associative but not commutative.
- **Invertible Function**: A function $f: X \to Y$ is said to be invertible if $\exists\, g: Y \to X$ such that $\begin{cases}(f \circ g)(y) = y \ \forall y \in Y \\ (g \circ f)(x) = x \ \forall x \in X\end{cases}$ If such a function g exists, then it is unique and is called the inverse of f. We denote the inverse of f (when it exists) by f^{-1}.
  - Theorem
    1. f is invertible $\Leftrightarrow$ f is bijection
    2. If $f: X \to Y$ is invertible, then f^{-1} is invertible, and $(f^{-1})^{-1} = f$.
    3. If $f: X \to Y$ and $g: B \to C$ are invertible, then $g \circ f$ is invertible, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

極限

- **定義**: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. We write $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.
  - 若極限值存在則唯一
- **單邊極限**
  - Left-hand Limit: $\lim_{x \to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$
  - Right-hand Limit: $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$
  - $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$
- **不存在的情況**
  - 左右極限不同 
  - 震盪 
  - Infinite Limit
    - 定義
      - $\lim_{x \to a} f(x) = \infty : \forall\, M > 0, \exists\, \delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > M$
      - $\lim_{x \to a} f(x) = -\infty : \forall\, N < 0, \exists\, \delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < N$
    - 垂直漸近線: The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true :
      $\lim_{x \to a} f(x) = \infty$ $\quad \lim_{x \to a^-} f(x) = \infty$ $\quad \lim_{x \to a^+} f(x) = \infty$
      $\lim_{x \to a} f(x) = -\infty$ $\quad \lim_{x \to a^-} f(x) = -\infty$ $\quad \lim_{x \to a^+} f(x) = -\infty$

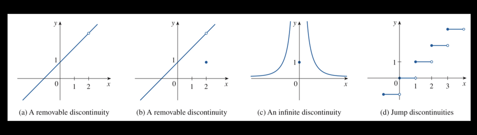


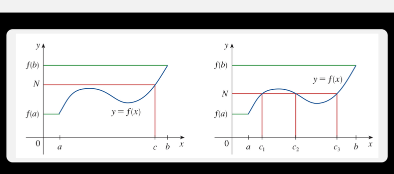
- **Proposition**
  - 極限的局部有限性: Suppose that $\lim_{x \to a} f(x) = L$, then $\exists\, \delta > 0,\ M > 0$ such that if $0 < |x - a| < \delta$ then $|f(x)| < M$.
  - If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limit exists.



  - Limit Laws: Suppose that c is a constant and the limits $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.
    - $\lim_{x \to a} [f(x) \pm g(x)] = L \pm M$
    - $\lim_{x \to a} [\, cf(x)\,] = cL$
    - $\lim_{x \to a} [f(x)g(x)] = LM$
    - $\lim_{x \to a} [f(x)]^n = L^n$ where $n \in \mathbb{N}$
    - $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \begin{cases} \frac{L}{M} \text{ if } M \neq 0 \text{ and } L \neq 0 \\ \text{doesn't exist if } M = 0 \end{cases}$
    - $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ where $n \in \mathbb{N}$. If n is even, we assume that $L > 0$.
    - 透過已知函數極限和 Limit Laws 可以更快地算出未知函數的極限！例如，透過 Limit Laws，我們可以知道 If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.
- **Theorem** — Squeeze Theorem: Let $p > 0$. Suppose that, for all x such that $f(x) \leq g(x) \leq h(x)$ when $0 < |x - c| < p$. If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.
- **Limit at Infinity**
  - 定義
    - $\lim_{x \to \infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that if $x > N$ then $|f(x) - L| < \varepsilon$
    - $\lim_{x \to -\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that if $x < N$ then $|f(x) - L| < \varepsilon$
    - 若極限存在則唯一！ All the Limit Laws are true when we replace $\lim_{x \to c}$ by $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.
  - Theorem
    1. If $r > 0$ is a rational number and $c \in \mathbb{R}$, then $\lim_{x \to \infty} \dfrac{c}{x^r} = 0$.
    2. If $r > 0$ is a rational number such that x^r is defined for all x and $c \in \mathbb{R}$, then $\lim_{x \to -\infty} \dfrac{c}{x^r} = 0$.
    - 這個定理在計算無窮遠處的極限時很好用，證明也不難，同學們可以自行嘗試證明。
  - 漸近線
    - Definition: The straight line $y = mx + k$ is an asymptote of the graph $y = f(x)$ if $\lim_{x \to \infty} (f(x) - mx - k) = 0$ or $\lim_{x \to -\infty} (f(x) - mx - k) = 0$. The straight line $y = mx + k$ is called a horizontal asymptote of the graph of f if $m = 0$, and is called a slant asymptote of the graph of f if $m \neq 0$.
      - 水平漸近線的定義可以寫成—The line $y = L$ is a horizontal asymptote of curve $y = f(x)$ if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.
    - 計算方法: If $y = mx + k$ is a slant asymptote of the graph $y = f(x)$, then $m = \lim_{x \to \infty} \dfrac{f(x)}{x}$ or $m = \lim_{x \to -\infty} \dfrac{f(x)}{x}$ and $k = \lim_{x \to \infty} (f(x) - mx)$ or $k = \lim_{x \to -\infty} (f(x) - mx)$.
      - 小技巧: 若曲線 $y = f(x)$ 可表為 $y = mx + k + g(x)$ 且 $\lim_{x \to \pm\infty} g(x) = 0$，則此曲線具有 $y = mx + k$ 的漸近線！
    - 觀念補充: 若 $\lim_{x \to \infty} f(x) = L$ 存在，則 $m = \lim_{x \to \infty} \dfrac{f(x)}{x} = 0$。若 $m = \lim_{x \to \infty} \dfrac{f(x)}{x}$ 存在且不等於 0，則 $\lim_{x \to \infty} f(x) = \pm\infty$。綜合上述，我們知道水平和斜漸近線不會並存，也就是只存在其中一個或都不存在。（上述的∞可改成$-\infty$）
  - Infinite Limit at Infinity
    - $\lim_{x \to \infty} f(x) = \infty : \forall\, M > 0, \exists\, N > 0$ such that if $x > N$ then $f(x) > M$
    - $\lim_{x \to \infty} f(x) = -\infty : \forall\, M < 0, \exists\, N > 0$ such that if $x > N$ then $f(x) < M$
    - $\lim_{x \to -\infty} f(x) = \infty : \forall\, M > 0, \exists\, N < 0$ such that if $x < N$ then $f(x) > M$
    - $\lim_{x \to -\infty} f(x) = -\infty : \forall\, M < 0, \exists\, N < 0$ such that if $x < N$ then $f(x) < M$
- **重要極限**
  - $\lim_{x \to a} 1 = 1$
  - $\lim_{x \to a} x = a$
  - $\lim_{x \to a} |x| = |a|$
  - $\lim_{x \to 0} \dfrac{\sin ax}{ax} = 1$ for each number $a \neq 0$
  - $\lim_{x \to \infty} \dfrac{\sin ax}{ax} = 0$ for each number $a \neq 0$
  - $\lim_{x \to 0} \dfrac{1 - \cos ax}{ax} = 0$ for each number $a \neq 0$
  - $\lim_{x \to \infty} \dfrac{\cos ax - 1}{ax} = 0$ for each number $a \neq 0$

連續

- **定義**
  - 在點上連續: A function f is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$
  - 單邊連續
    - A function f is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$
    - A function f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$
    - f is continuous at a iff $f(a)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$ all exist and are equal.
  - 在區間上連續: A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)
  - 處處連續: A function f is continuous everywhere if it is continuous at its domain.
- **不連續的情況** 
- **Proposition**
  - If f and g are continuous at a and c is a constant, then the following functions are also continuous at a : 1. $f \pm g$ 2. cf 3. fg 4. $\dfrac{f}{g}$ if $g(a) \neq 0$
  - If f is continuous at b, and $\lim_{x \to a} g(x) = b$ then $\lim_{x \to a} f(g(x)) = f(b)$. In other words, $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$
    - Corollary: If f is continuous at $g(a)$ and g is continuous at a $\left(\lim_{x \to a} g(x) = g(a)\right)$, then $\lim_{x \to a} f(g(x)) = f(g(a))$. That is $f(g(x))$ is continuous at a.
- **Theorem**
  - Intermediate Value Theorem: Suppose that g is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



    - Special Case — Balzano's Theorem 勘根定理: If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.
  - Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval. (連續函數在閉區間上必有最大最小值)
- **在定義域上一定連續的函數**
  - Polynominal Function
  - Rational Function
  - Root Function
  - Logarithmic Function
  - Exponential Function
  - Trigonometric Function
  - Inverse Trigonometric Function