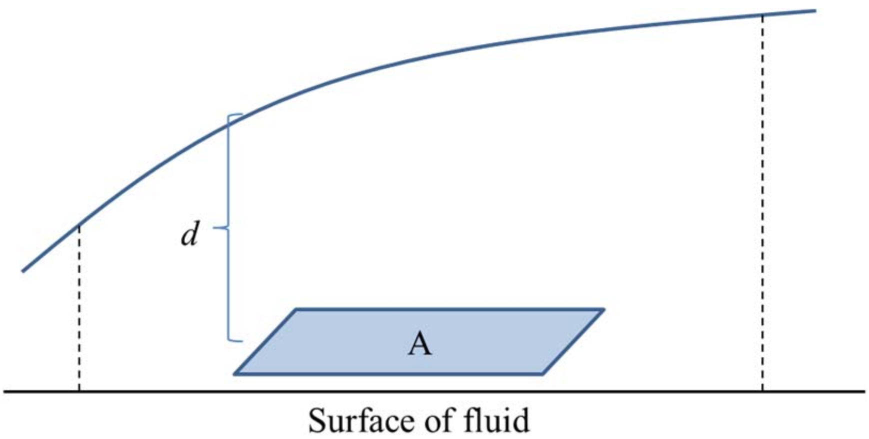
§8-3 Applications to Physics and Engineering

I. Hydrostatic Pressure and Force



$$p=\frac{F}{A}=\frac{mg}{A}=\frac{\rho Ad}{A}=\rho gd.$$

F : force m : mass g : gravity accelereation

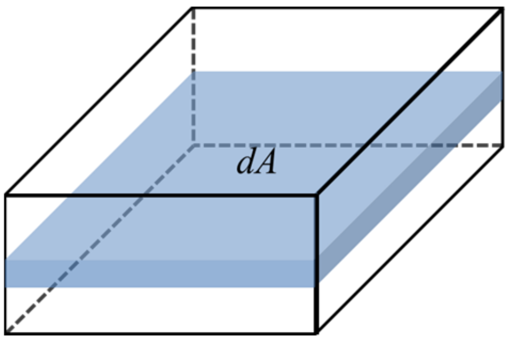
ρ : density of a fluid in metric system

δ : density of a fluid in $ft-pound$ system

Water density : $1000\,{}^{kg}\!/\!{}_{m^3}$ or $62.5\,{}^{pd}\!/\!{}_{ft^3}$ (含重力)

(1) $P(\text{pressure})=\begin{cases}\rho gd & \text{(metric system)}\\ \delta d & (ft-pound\ \text{system})\end{cases}$

(2) Force against a vertical plate or wall or dam.

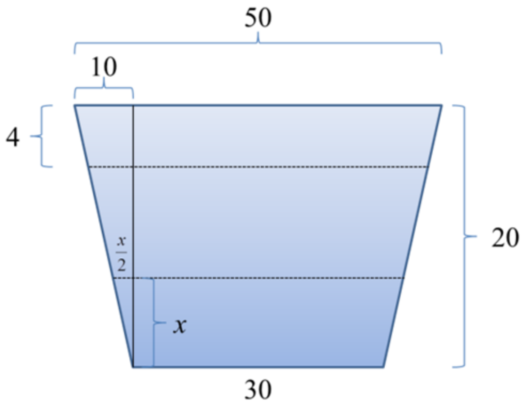


$F=\int \rho dA$ (ρ, A 隨位置而變)

Example 1：A dam has the shape of the trapezoid. The height is 20m, and which is

50*m* at the top and 30*m* at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4*m* from the top of the dam.

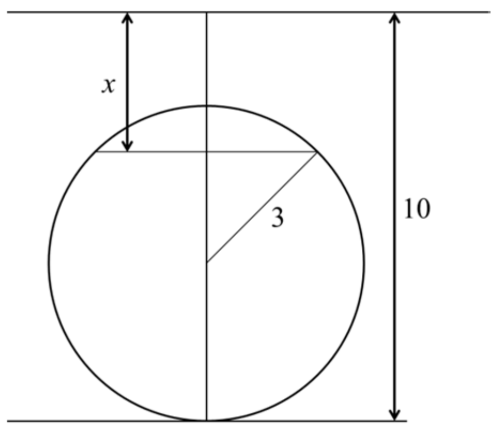
Solution：



$$F = \int_0^{16} \rho g (16-x)(30+x)dx$$

Example 2：Find the hydro static force on one end of a cylindrical drum with radius 3*ft* if the drum is submerged in water 10*ft* deep.

Solution：



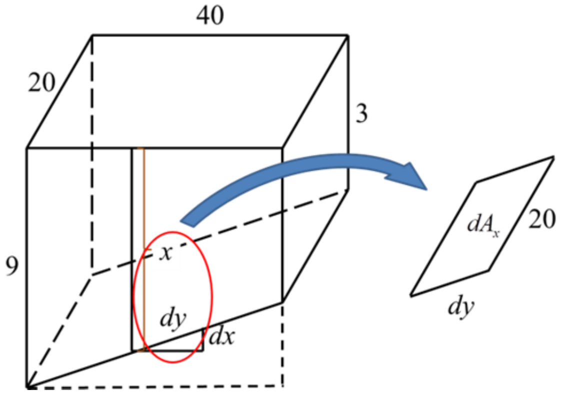
$$dA = 2\sqrt{9-(7-x)^2}\,dx$$

$$F = 2\int_4^{10} 62.5x\sqrt{9-(7-x)^2}\,dx$$

$$\overset{7-x=y}{\Rightarrow} \int_{-3}^{3} 62.5(7-y)\sqrt{9-y^2}\,dy.$$

Example 3：Force against an inclined bottom.

Solution：



$$\int_3^9 \rho dA_x$$

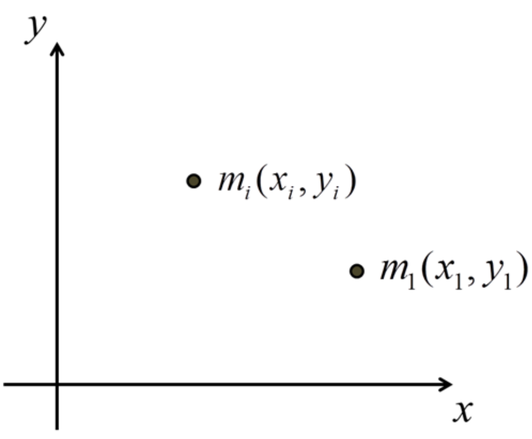
$$= \int_3^9 \delta dA_x$$

$$= \int_3^9 \delta \times 20x dy$$

$$\frac{dy}{dx} = \frac{\sqrt{1636}}{6} = \frac{\sqrt{409}}{3} \Rightarrow dy = \frac{\sqrt{409}}{3} dx$$

$$= \int_3^9 \frac{20\sqrt{409}}{3} \delta x dx.$$

∗ Center $(\bar{x}, \bar{y})$ of a discrete system.

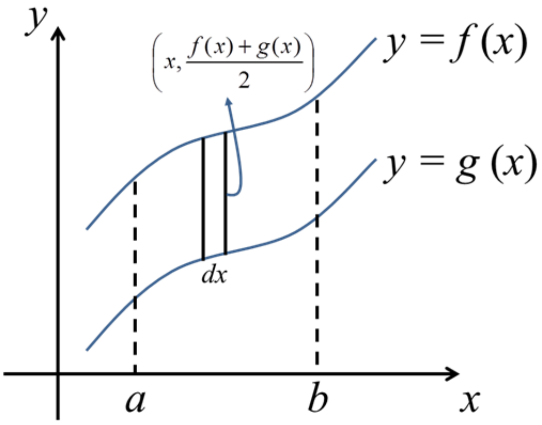


$$M_y = \sum_{i=1}^{n} m_i x_i$$

$$M_x = \sum_{i=1}^{n} m_i y_i$$

$$\Rightarrow \bar{x} = \frac{M_y}{\sum_{i=1}^{n} m_i}, \quad \bar{y} = \frac{M_x}{\sum_{i=1}^{n} m_i}$$

$*$Center $(\bar{x}, \bar{y})$ of a continuous system with uniform density ρ.



$\left(x, \frac{f(x)+g(x)}{2}\right)$: center of infinitesimal strip at x (the shaped part on the right).

$$M_x = \int_a^b \frac{(f(x)+g(x))}{2} \times \rho(f(x)+g(x)) = \rho\int_a^b \frac{(f^2(x)+g^2(x))}{2} dx.$$

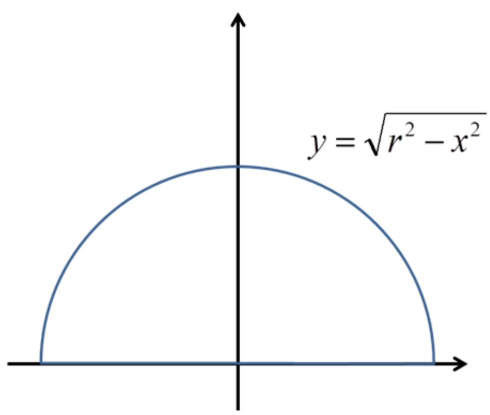
$$M_y = \int_a^b x \times \rho(f(x)+g(x)) = \rho\int_a^b x(f(x)-g(x))dx.$$

$$\Rightarrow \bar{y} = \frac{\frac{1}{2}\int_a^b (f^2(x)-g^2(x))dx}{\int_a^b (f(x)-g(x))dx}$$

$$\bar{x} = \frac{\int_a^b x(f(x)-g(x))dx}{\int_a^b (f(x)-g(x))dx}.$$

Example 4：Find the center of a semicircular plate with radius r.

Solution：

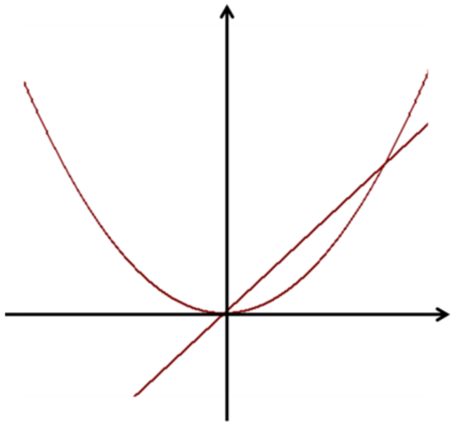


$$\Rightarrow \bar{y} = \frac{\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx}{\frac{1}{2}\pi r^2} = \frac{\int_{-r}^{r}\left(r^2-x^2\right)dx}{\pi r^2}$$

$$= \frac{1}{\pi r^2}\frac{1}{2}\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} = \frac{4r}{3\pi}.$$

Example 5：Find the centroid of the region bounded by $y = x$ and $y = x^2$.

Solution：



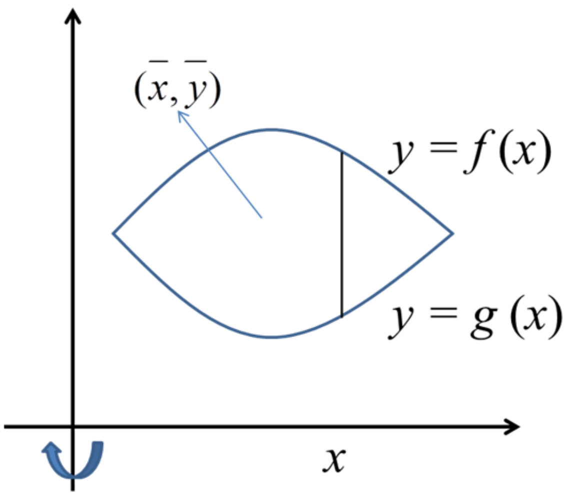
$$A = \int_0^1 \left(x - x^2\right) dx = \frac{1}{6}$$

$$M_x = \int_0^1 \left(x^2 + x^4\right) dx = \frac{1}{15}$$

$$M_y = \int_0^1 x\left(x - x^2\right) dx = \frac{1}{12}$$

$$\Rightarrow \bar{x} = \frac{1}{2},\quad \bar{y} = \frac{2}{5}.$$

Theorem of Pappu：



$$V=\int_a^b 2\pi x\left(f(x)-g(x)\right)dx$$

$$=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$=2\pi(\bar{x}A)$$

$$=(2\pi\bar{x})A$$

$$=\left(\text{distance traveld by the region}\right)\times\left(\text{the area of the region}\right).$$