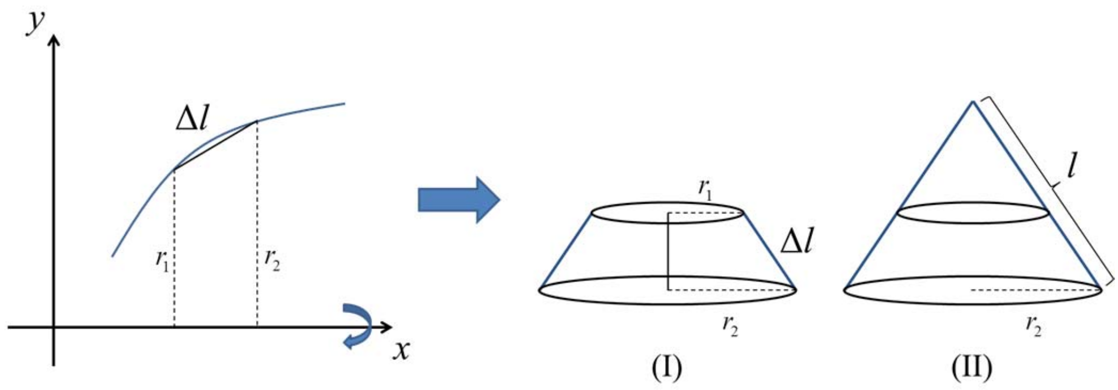
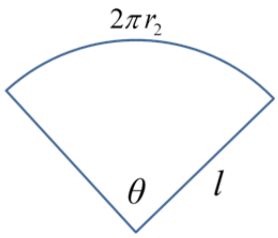
# §8-2 Area of a Surface Revolution

＊公式：**Area of a Surface Revolution = *S***




(II) 圖之表面積=




$$=\frac{1}{2}l^2\theta=\frac{1}{2}l^2\frac{2\pi r_2}{l}=\pi l r_2.$$

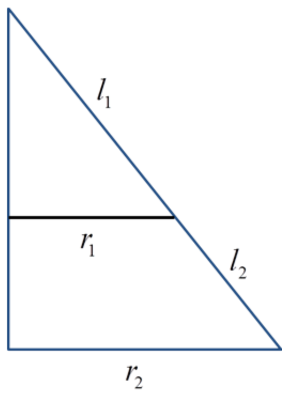
$$=\pi\left(l_1+l_2\right)r_2-\pi l_1 r_1$$

$$=\pi\left[\left(r_2-r_1\right)l_1+r_2 l_2\right]$$

$$\Rightarrow \frac{l_1}{l_1+l_2}=\frac{r_1}{r_2}$$

$$\Rightarrow \left(r_2-r_1\right)l_1=r_1 l_2.$$

(I) 圖之表面積=



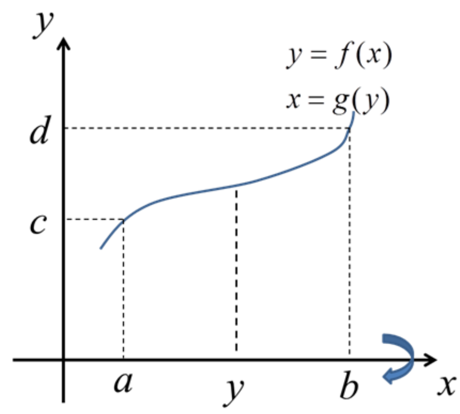
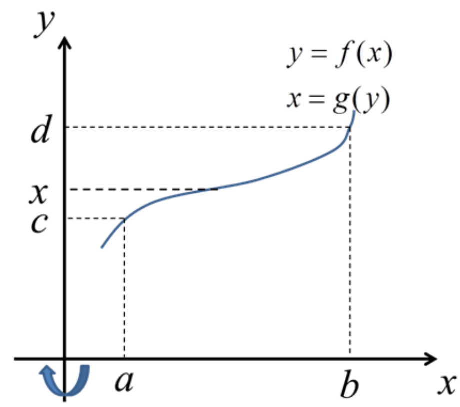

$$\therefore \text{(I)圖之表面積}=\pi(r_1+r_2)l_2$$

$$=\pi(r_1+r_2)ds$$

$$=2\pi r ds$$

$$r=\frac{r_1+r_2}{2}(=y=f(x)\text{當}dx\text{是無窮小時}).$$

$$\Rightarrow S=\int 2\pi r ds.$$

| | |
|---|---|
| $S=\int 2\pi r\,ds=\begin{cases}\int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ \int_c^d 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy.\end{cases}$ |  |
| $S=\int 2\pi r\,ds=\begin{cases}\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ \int_c^d 2\pi g(y)\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy.\end{cases}$ |  |

**Example 1：**

Find the surface area of $y=\sqrt{r^2-x^2}$, $-r\le x\le r$ rotating about the $x$-axis.

**Solution：**

$$y'=(-x)\left(r^2-x^2\right)^{\frac{1}{2}}\Rightarrow 1+(y')^2=\frac{r^2}{r^2-x^2}$$

$$S=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$=2\pi r\int_{-r}^{r}dx$$

$$=4\pi r^2.$$

**Example 2：**

Find the surface area of $y = x^2$ from (1,1) to (2,4), rotating about the $y$-axis.

**Solution：**

$$x = y^{\frac{1}{2}}$$

$$\Rightarrow x' = \frac{1}{2} y^{-\frac{1}{2}}$$

$$\Rightarrow 1 + (x')^2 = \frac{1+4y}{4y}$$

$$S = \int_1^4 2\pi y^{\frac{1}{2}} \sqrt{\frac{1+4y}{4y}} dy$$

$$= \pi \int_1^4 \sqrt{1+4y} dy$$

$$= 2\pi \frac{2}{3} \left( y + \frac{1}{4} \right)^{\frac{3}{2}} \Bigg|_1^4 = \frac{\pi}{6} \left( 17\sqrt{17} - 5\sqrt{5} \right).$$

**Alternate solution：**

$$y' = 2x \Rightarrow 1 + (y')^2 = 1 + 4x^2$$

$$S = \int_1^2 2\pi x \sqrt{1+4x^2} dx$$

$$= \frac{\pi}{6} \left( 17\sqrt{17} - 5\sqrt{5} \right).$$



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**Example 3：**

Find the surface area of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, rotating about the $x$-axis.

**Solution：**

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$\Rightarrow y'=\frac{b}{a}(-x)\left(a^2-x^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow 1+(y')^2=\frac{b^2x^2+a^4-a^2x^2}{a^2\left(a^2-x^2\right)}$$

$$S=2\int_0^a 2\pi\frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-\left(a^2-b^2\right)x^2}}{a\sqrt{a^2-x^2}}dx$$

$$=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-\left(a^2-b^2\right)x^2}\,dx.$$

$$x=\frac{a^2}{\sqrt{a^2-b^2}}\sin\theta$$

$$=\frac{4\pi b}{a^2}\int_0^\alpha a^2\cos\theta\frac{a^2}{\sqrt{a^2-b^2}}\cos\theta d\theta$$

$$=\frac{4\pi a^2b}{\sqrt{a^2-b^2}}\left(\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\right)\Bigg|_0^\alpha$$

$$=\frac{4\pi a^2b}{\sqrt{a^2-b^2}}\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}+\frac{b\sqrt{a^2-b^2}}{2a^2}\right)$$

$$=2\pi\left[b^2+\frac{a^2b}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right].$$