§7-8 Improper Integrals

＊What is the improper integral $\int_a^b f(x)dx$?

1. $a=\pm\infty$ or $b=\pm\infty$.
2. $f(c)=\pm\infty$ for some $c\in[a,b]$.

發生以上 1.或 2.情況，即可稱$\int_a^b f(x)dx$ 為 improper integral.

＊幾何上來說，improper integral 所圍出來的區域是無界的。

(1−a) $\int_a^{\infty} f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$

(1−b) $\int_{-\infty}^{\infty} f(x)dx=\underbrace{\int_{-\infty}^{0} f(x)dx}_{(1-a)}+\underbrace{\int_0^{\infty} f(x)dx}_{(1-a)}$

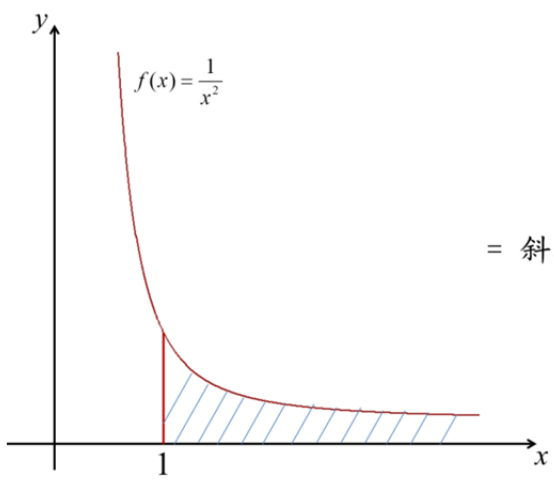
(2−a) $c=a$ or b

$\int_a^b f(x)dx=\lim_{d\to a^+}\int_d^b f(x)dx$ or $\int_a^b f(x)dx=\lim_{d\to b^-}\int_a^d f(x)dx$

(2−b) $a<c<b$

$\int_a^b f(x)dx=\lim_{d\to c^-}\int_a^d f(x)dx+\lim_{d\to c^+}\int_d^b f(x)dx$

＊c means the discontinuous points.

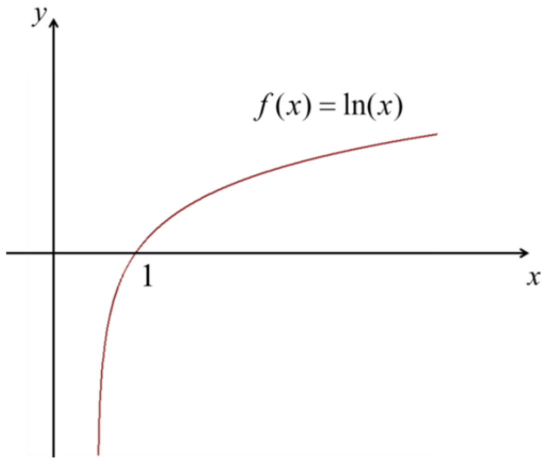


= 斜線區域(無界)的面積

＊一個 improper Integral 可能有好幾個缺點

$$\int_0^\infty \ln x dx = \int_0^1 \ln x dx + \int_1^\infty \ln x dx$$

$$= \lim_{b\to 0^+}\int_b^1 \ln x dx + \lim_{c\to\infty}\int_1^c \ln x dx$$



＊註記：

1. 要計算具有多個缺點的 improper integral，一定要將原 improper integral 表成多個新的 improper integrals 的組合，其中每一個新的 improper integral 都只能有一個缺點。

2. 如果每一個只有一個缺點的 improper integral 所代表的無界區域之面積皆為有界，即原 improper integral 稱之為收斂 (convergence)，反之則稱為發散(divergence)。

Example 1： Verify the convergence of $\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx.$

Solution：

$$\int_1^b \frac{1}{x^p}dx = \begin{cases} \left.\dfrac{x^{-p+1}}{-p+1}\right|_1^b & p \neq 1 \\ \ln|x|\,\big|_1^b & p = 1 \end{cases}$$

$$\Rightarrow \int_1^\infty \frac{1}{x^p}dx = \begin{cases} \text{convergence} & p > 1 \\ \text{divergence} & p \le 1 \end{cases}.$$

Example 2： Verify the convergence of $\int_0^1 \frac{1}{x^p}dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{x^p}$.

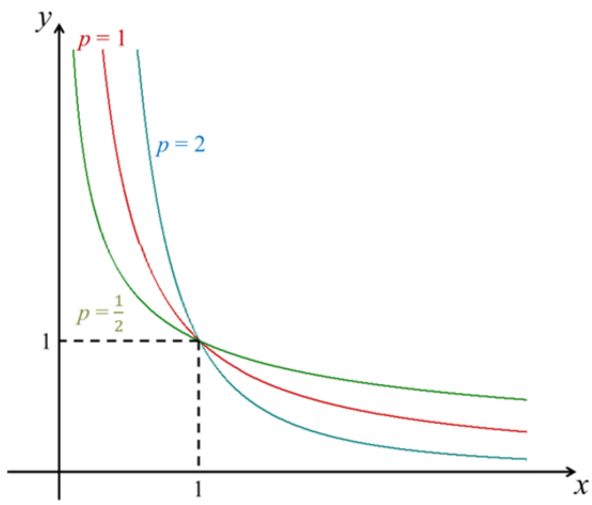
Solution：

$$\int_0^1 \frac{1}{x^p}dx = \begin{cases} \text{convergence} & p<1 \\ \text{divergence} & p\geq 1 \end{cases}.$$

＊Summary：

i. Example 1 中，p 要大(即 $f(x)=\frac{1}{x^p}$ 跑到 0 的速度要夠快，當 $x\to\infty$ 時)，無窮區域的面積才可能是有限。

ii. Example 2 中，p 要小(即 $f(x)=\frac{1}{x^p}$ 跑到 ∞ 的速度要夠慢，當 $x\to 0$ 時)，無窮區域的面積才有可能有限。



＊Comparison Theorem： $f\geq g\geq 0$

i. $\int_a^b g(x)dx$: diverges $\Rightarrow \int_a^b f(x)dx$: diverges.

ii. $\int_a^b f(x)dx$: converges $\Rightarrow \int_a^b g(x)dx$: converges.

Example 3： Verify the convergence of $\int_{-\infty}^{0} xe^{x}\,dx$.

Solution：

Converges.

Example 4： Verify the convergence of $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution：

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1} x\Big|_{-\infty}^{0} + \tan^{-1} x\Big|_{0}^{\infty}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Example 5： Verify the convergence of $\int_{0}^{3} \frac{1}{x-1}\,dx$.

Solution：

$$\int_{0}^{3} \frac{1}{x-1}\,dx$$

$$= \underbrace{\int_{0}^{1} \frac{1}{x-1}\,dx}_{\text{diverges}} + \underbrace{\int_{1}^{3} \frac{1}{x-1}\,dx}_{\text{diverges}}.$$

Example 6： Verify the convergence of $\int_0^1 \ln x dx$.

Solution：

$$\int_0^1 \ln x dx$$

$$= x\ln x\Big|_0^1 - x\Big|_0^1 = -1$$

$*$ 另解：$-\ln x$比$\dfrac{1}{x^{\frac{1}{2}}}$跑到$\infty$的速度慢，當$x \to 0$時$\Rightarrow$ converges.

Example 7： Verify the convergence of $\int_2^\infty \frac{1}{\ln x} dx$.

Solution：

$\because \dfrac{1}{x} < \dfrac{1}{\ln x}$當$x$很大時$\Rightarrow$ diverges.

Example 8： Verify the convergence of $\int_0^\infty e^{-x^2} dx$.

Solution：

$e^{x^2} > x^2 \Rightarrow e^{-x^2} < x^{-2} \Rightarrow$ converges.(compare with $\dfrac{1}{x^2}$)

Note that e^(-x^2)is one when evaluating at zero. Hence, nothing is improper at zero. To verify the convergence of the integral of e^(-x^2) from 0 to infinity, one only needs to check how quickly the speed of the function e^(-x^2) goes to zero at infinity.

In fact, it goes zero much quicker than that of x^(-2) at infinity. Hence, such improper integral converges.

Example 9： Verify the convergence of $\int_1^\infty \frac{1+e^{-x}}{x} dx$.

Solution：

$\dfrac{1+e^{-x}}{x} > \dfrac{1}{x} \Rightarrow$ diverges.(compare with $\dfrac{1}{x}$)

Example 10： Verify the convergence of $\int_0^\infty x^2 e^{-x^2}\,dx$.

Solution：

$$u = x \qquad dv = xe^{-x^2}\,dx$$
$$du = dx \qquad v = -\frac{1}{2}e^{-x^2}$$

$$\int_0^\infty x^2 e^{-x^2}\,dx$$

$$= -\frac{1}{2}xe^{-x^2}\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x^2}\,dx.$$

(和$\frac{1}{x^2}$比較可得此瑕積分為收斂)

歷屆大會考考古題精選：

1. If n us a positive integer, then $\int_0^\infty x^n e^{-x}\,dx = ?$
(A) 1; (B) n; (C) $n!$; (D) ∞.

2. Evaluate $\int_1^\infty \frac{\ln x}{x^2}\,dx =$ ___________.

3. The value of $\int_0^\infty \frac{\ln x}{1+x^2}\,dx$ is ___________.

4. Evaluate the improper integral $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx =$ ___________.

5. Evaluate $\int_{-\infty}^\infty \frac{dx}{4x^2+4x+5} =$ ___________.