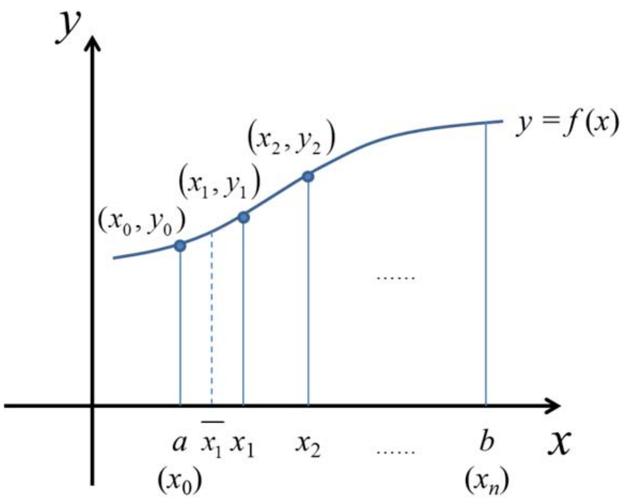
§7-7 Approximate Integration



$$\overline{x_i}=\frac{x_{i-1}+x_i}{2} \qquad \Delta x=\frac{b-a}{n}$$

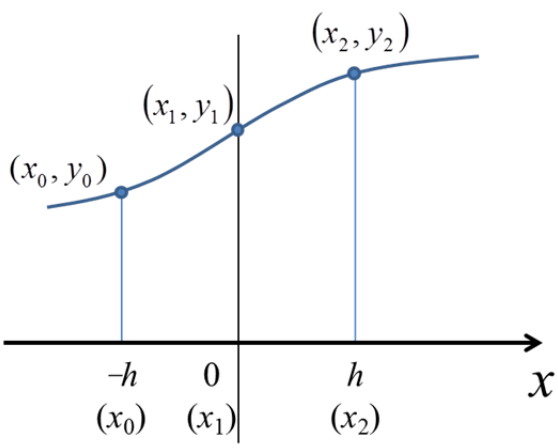
1. Midpoint Rule：(小長方形的高 ＝ 小區間中點的 f 值)

$$\int_a^b f(x)dx \approx \Delta x \times \left(f\left(\overline{x_1}\right)+\ldots+f\left(\overline{x_n}\right)\right).$$

2. Trapezoidal Rule：(每小區間用梯形來逼近)

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n).$$

3. Simpson's Rule：(二小區間的 f 圖上三點用一拋物線來逼近)



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找一拋物線Γ：$y = ax^2 + bx + c$ 經過$(-h, y_0), (0, y_1), (h, y_2)$

$$\Rightarrow \left.\begin{aligned} y_0 &= ah^2 - bh + c \\ y_1 &= c \\ y_2 &= ah^2 + bh + c \end{aligned}\right\} \Rightarrow y_0 + y_2 \Rightarrow y_0 + 4y_1 + y_2 = 2ah^2 + 6c$$

$$\Rightarrow \int_{-h}^{h}\left(ax^2 + bx + c\right)dx = 2\int_{0}^{h}\left(ax^2 + c\right)dx$$

$$= \frac{h}{3}\left(2ah^2 + 6c\right)$$

$$= \frac{h}{3}\left(y_0 + y_2 + 4y_1\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

也即

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{n-1} + y_n\right) \quad (n \text{ must be even})$$

∗How accurate those approximation are?

$$\text{誤差} = \begin{cases} E_M \le \dfrac{k(b-a)^3}{24n^2}, & k = \max\limits_{a\le x\le b}\left|f^{(n)}(x)\right|. \\ E_T \le \dfrac{k(b-a)^3}{12n^2}. & \\ E_S \le \dfrac{\bar{k}(b-a)^5}{180n^4}, & \bar{k} = \max\limits_{a\le x\le b}\left|f^{(4)}(x)\right|. \end{cases}$$

Example 1： Evaluate $\int_1^2 \frac{1}{x}\,dx$ by midpoint rule, n = 5.

Solution：

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx \approx 0.2 \times \left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)$$

$$\approx 0.691908$$

$$\Rightarrow \ln 2 \approx 0.691908.$$

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