

§7-2 Trigonometric Integrals

*題型：

I. $\int \sin^m x \times \cos^n x dx.$

II. $\int \tan^m x \times \sec^n x dx, \int \cot^m x \times \csc^n x dx.$

III. $\int \sin mx \times \cos n x dx, \int \sin mx \times \sin n x dx, \int \cos mx \times \cos n x dx.$

I.

1. m or n is odd :

● $\begin{cases} u = \sin x \\ u = \cos x \end{cases} \Rightarrow \begin{cases} du = \cos x dx \\ du = -\sin x dx \end{cases}$

● $\cos^2 x + \sin^2 x = 1.$

● 選擇偶數次方為 u ，若 m 、 n 皆為奇數，則 u 可任意選擇。

$$\int \sin^3 x \cos^2 x dx = \int \sin^2 x \cos^2 x \sin x dx = \int (1-u^2)u^2(-du).$$

$$\int \cos^5 x \sin^4 x dx = \int \cos^4 x \sin^4 x \cos x dx = \int (1-u^2)^2 u^4 du.$$

$$\int \cos^3 x \sin^3 x dx = \begin{cases} \int \cos^3 x \sin^2 x \sin x dx = \int u^3(1-u^2)(-du). & (u = \cos x) \\ \int \cos^2 x \sin^3 x \cos x dx = \int (1-u^2)u^3 du. & (u = \sin x) \end{cases}$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \ln |\sec x| + c.$$

2. m and n are even :

利用 $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$, $\sin^2 \theta = \frac{1-\cos 2\theta}{2}$ 降階

$$\int \sin^4 \theta d\theta = \int \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4} \int (1 - 2\cos 2\theta + \cos^2 2\theta) d\theta$$

$$\frac{1}{4} \theta - \frac{1}{4} \sin 2\theta + \frac{1}{4} \int \frac{1 + \cos 4\theta}{2} d\theta = \frac{3\theta}{8} - \frac{1}{4} \sin 2\theta + \frac{1}{32} \sin 4\theta + c.$$

II.

- $\begin{cases} u = \tan x \\ u = \sec x \end{cases} \Rightarrow \begin{cases} du = \sec^2 x dx \dots\dots (a) \\ du = \sec x \tan x dx \dots (b) \end{cases}$

- $\sec^2 x = \tan^2 x + 1.$

1. n is even : 利用 (a)

$$\int \tan^3 x \sec^4 x dx$$

$$= \int \tan^2 x \sec^2 x \sec^2 x dx = \int u^3 (1 + u^2) du.$$

2. m is odd : 利用 (b)

$$\int \tan^3 x \sec^3 x dx$$

$$= \int \tan^2 x \sec^2 x \times \tan x \sec x dx = \int (u^2 - 1) u^2 du.$$

3. m is even and n is odd :

(i). $\int \sec x dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \ln |\sec x + \tan x| + c.$

(ii). $\int \sec^3 x dx$

$$\begin{cases} u = \sec x \\ dv = \sec^2 x \end{cases} \Rightarrow \begin{cases} du = \sec x \tan x dx \\ v = \tan x \end{cases}$$

$$\Rightarrow \int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$\Rightarrow \int \sec^3 x dx = \frac{1}{2} [\sec x \tan x + \ln |\tan x + \sec x|] + c.$$

III. 將乘法變加法

$$1. \sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)].$$

$$2. \sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)].$$

$$3. \cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)].$$