

§7-1 Integration by Parts

* 公式 : Integration by parts

- $\int u dv = uv - \int v du.$
- $\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$

* Integration by parts(積分) \leftrightarrow Product rule(微分)

$$u = f(x), v = g(x);$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow \int \underbrace{(f(x)g(x))'}_{uv} dx = \int \underbrace{g(x)}_v \underbrace{f'(x)}_{du} dx + \int \underbrace{f(x)}_u \underbrace{g'(x)}_{dv} dx$$

$$\Rightarrow uv = \int v du + \int u dv$$

$$\Rightarrow \int u dv = uv - \int v du.$$

* Keys :

1. Identify u and dv .
2. Easy to recover v from dv .
3. $\int v du$ is not complicated than $\int u dv$.

Example 1 :

$$\int x \sin x dx = ?$$

Solution :

$$\begin{cases} u = x \\ dv = \sin x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = -\cos x \end{cases}$$

$$\Rightarrow \int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + c.$$

Example 2 :

$$\int \tan^{-1} x dx = ?$$

Solution :

$$\begin{cases} u = \tan^{-1} x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{dx}{1+x^2} \\ v = x \end{cases}$$

$$\Rightarrow \int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c.$$

Example 3 :

$$\int \ln(x-1)dx = ?$$

Solution :

$$\begin{cases} u = \ln(x-1) \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{dx}{x-1} \\ v = x-1 \end{cases}$$

$$\Rightarrow \int \ln(x-1)dx = (x-1)\ln(x-1) - \int 1dx = (x-1)\ln(x-1) - x + c.$$

Example 4 :

$$\int x^2 e^x dx = ?$$

Solution :

$$\begin{cases} u = x^2 \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2x dx \\ v = e^x \end{cases}$$
$$\Rightarrow \int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

$$\begin{cases} u = 2x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2 dx \\ v = e^x \end{cases}$$

$$\int x^2 e^x dx = x^2 e^x - \left(2x e^x - 2 \int e^x dx \right) = x^2 e^x - 2x e^x + 2e^x + c$$

Example 5 :

$$\int e^x \sin x dx = ?$$

Solution :

$$\begin{cases} u = \sin x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = \cos x dx \\ v = e^x \end{cases}$$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

$$\begin{cases} u = \cos x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = -\sin x dx \\ v = e^x \end{cases}$$

$$\int e^x \sin x dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x dx \right)$$

$$\Rightarrow \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

Example 6 :

$$\int \cos \sqrt{x} dx = ? \text{ (Hint : Substitution + Integration by parts)}$$

Solution :

$$\text{Let } u = \sqrt{x} \Rightarrow du = \frac{1}{2} x^{-\frac{1}{2}} dx$$

$$\Rightarrow 2 \int u \cos u du$$

$$\begin{cases} v = u \\ dw = \cos u du \end{cases} \Rightarrow \begin{cases} dv = du \\ w = \sin u \end{cases}$$

$$\Rightarrow 2 \left(u \sin u - \int \sin u du \right)$$

$$= 2u \sin u + 2 \cos u + c$$

$$= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + c.$$

Example 7 :

$$\int_0^1 x^3 e^{-x^2} dx = ?$$

Solution :

$$\text{Let } u = x^2 \Rightarrow du = 2x dx$$

$$\Rightarrow \frac{1}{2} \int_0^1 u e^{-u} du = \frac{1}{2} \left(-u e^{-u} - e^{-u} \Big|_0^1 \right) = \frac{1}{2} [(-e^{-1} - e^{-1}) + 1] = -e^{-1} + \frac{1}{2}.$$