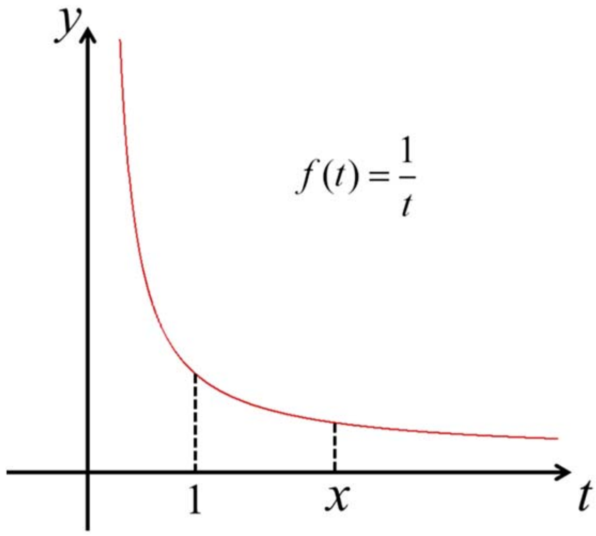
§5-6 The Logarithm Defined as an Integral

＊Definition：



$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x>0$$

I. Properties of ln x.

(i) ln 1 = 0, ln x > 0, if x > 1; ln x < 0 if x < 1.

(ii) $\frac{d}{dx} = \ln x = \frac{1}{x}$.

(iii) $\ln xy = \ln x + \ln y$.

Proof of (iii)

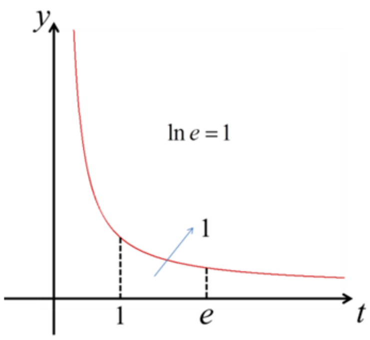
Let $f(x)=\ln ax, \quad a>0 \Rightarrow f(x)=\int_1^{ax}\frac{1}{\tau}d\tau$

$\Rightarrow f'(x)=\frac{1}{ax}, \quad a>0 \Rightarrow \ln ax = \ln x + C$

$\overset{pick\ x=1}{\Rightarrow} \ln a = \ln 1 + C \Rightarrow C = \ln a$

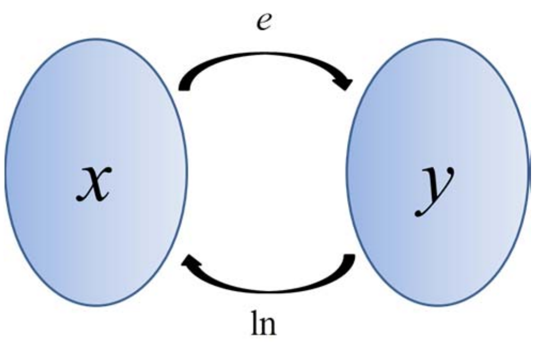
$\Rightarrow \ln ax = \ln x + \ln a.$

II. Define e to be such that



III. Exponential Function： $e^x = y \Leftrightarrow \ln y = x$

(inverse function of ln)



$$\ln y = x \Rightarrow \frac{1}{y} y' = 1 \Rightarrow y' = y \Rightarrow \frac{d}{dx} e^x = e^x.$$

IV.

(i) General Exponential Functions.

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a} \Rightarrow \frac{d}{dx} a^x = (\ln a) a^x.$$

(ii) General Logarithm (inverse function of $f(x) = a^x$)

$$\log_a x = y \Leftrightarrow a^y = x \Rightarrow (\ln a) a^y y' = 1$$
$$\Rightarrow y' = a^y \left(\frac{1}{\ln a}\right) \Rightarrow \frac{d}{dx} \log_a x = \left(\frac{1}{\ln a}\right) x.$$

V. $e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}.$

Remark：

有些微積分的書先講積分，再提微分。如此則可先定義 $y=\ln x$，再定義 $y=e^x$，再定義 $y=\log_a x$。這樣的好處可以避免需要解釋什麼是 2^π？

Example 1：$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\ln n<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}.$

Solution：