§5-5 The Substitution Rules

＊Two systematic methods to do integration：

1. The substitution rule (積分) ⇔The Chain rule (微分).

2. Integration by parts (積分) ⇔The Product rule (微分).

Example 1： $\int 2x\sqrt{1+x^2}\,dx = ?$

＊**Key**：Just like doing differentiation by chain rule, we first need to identify *u*.

Solution：

Let $u = 1 + x^2$

$$\left.\begin{aligned} \frac{du}{dx} &= 2x \\ du &= 2xdx \end{aligned}\right\} \Rightarrow \int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}.$$

Example 2： $\int \frac{x}{\sqrt{1-4x^2}}\,dx = ?$

Solution：

Let $u = 1 - 4x^2$

$$\left.\begin{aligned} \frac{du}{dx} &= -8x \\ -\frac{1}{8}du &= xdx \end{aligned}\right\} \Rightarrow -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du = -\frac{1}{4}u^{\frac{1}{2}} + C = -\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}} + C.$$

＊**Summary**：

$$\int f(g(x)) \times g'(x)dx = \int f(u)du$$

$$\begin{array}{cc} \downarrow & \downarrow \\ u & du \end{array}$$

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Example 3：$\int x^3 \cos\left(x^4+2\right)dx = ?$

Solution：

Let $u = x^4 + 2$

$$\left.\begin{array}{l}\dfrac{du}{dx} = 4x^3 \\ \dfrac{1}{4}du = x^3 dx\end{array}\right\} \Rightarrow \frac{1}{4}\int \cos u du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin\left(x^4+2\right) + C.$$

Example 4：$\int \dfrac{x}{\sqrt{x-1}}dx = ?$

Solution：

Let $u = x - 1 \Rightarrow x = u + 1$

$$du = dx$$

$$\Rightarrow \int \frac{(u+1)}{\sqrt{u}}dx = \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C.$$

Example 5：$\int \tan x dx = \int \dfrac{\sin x}{\cos x}dx = ?$

Solution：

Let $u = \cos x$

$$du = -\sin x dx$$

$$\Rightarrow -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = -\ln|\sec x| + C.$$

Example 6：$\int_1^e \frac{\ln x}{x} dx = ?$

Solution：

Let $u = \ln x$

$$du = \frac{1}{x} dx \Rightarrow \int_0^1 u du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Example 7：$\int_0^4 \sqrt{2x+1} dx = ?$

Solution：

Let $u = 2x+1$

$$du = 2dx$$

$$\Rightarrow \frac{1}{2}\int_1^9 u^{\frac{1}{2}} du = \frac{1}{3} u^{\frac{3}{2}}\Big|_1^9 = 9 - \frac{1}{3} = \frac{26}{3}.$$

Example 8：$\int \sec^3 x \times \tan x dx = ?$

Solution：

Let $u = \sec x$

$$du = \sec x \times \tan x dx$$

$$\Rightarrow \int \sec^2 x \times (\sec x \times \tan x) dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C.$$

Example 9：$\int \frac{x}{\sqrt[4]{x+2}}\,dx = ?$

Solution：

Let $u = x + 2$

$du = dx$

$\Rightarrow \int \frac{u-2}{u^{\frac{1}{4}}}\,du = \int u^{\frac{3}{4}} - 2u^{-\frac{1}{4}}\,du = \frac{4}{7}(x+2)^{\frac{7}{4}} - \frac{8}{3}(x+2)^{\frac{3}{4}} + C$

Example 10：$\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx = ?$

Solution：

Let $u = 2x$

$du = 2dx$

$\Rightarrow \int_0^2 f(2x)\,dx = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}\int_0^4 f(x)\,dx = 5.$

Example 11：Prove that $\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

(Hint：$u = \pi - x$.)

Solution：

Let $u = \pi - x$

$du = -dx$

$\Rightarrow \int_0^{\pi} xf(\sin x) = -\int_{\pi}^{0} (\pi - u) f(\sin(\pi - u))du$

$= \int_0^{\pi} (\pi - u) f(\sin u)du$

$= \pi\int_0^{\pi} f(\sin u)du - \int_0^{\pi} uf(\sin u)du$

$= \pi\int_0^{\pi} f(\sin u)du - \int_0^{\pi} xf(\sin x)dx$

That is, $\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx$

$$\Rightarrow \int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$$

Example 12：Use example 11 to evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = ?$

$$\left(\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx\right)$$

Solution：

Let $f(y) = \frac{y}{2-y^2} \Rightarrow f(\sin x) = \frac{\sin x}{2-\sin^2 x} = \frac{\sin x}{1+\cos^2 x}$

$$\Rightarrow \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx \overset{ex.11}{=} \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Let $u = \cos x$

$du = -\sin x dx$

$$\Rightarrow -\frac{\pi}{2}\int_1^{-1} \frac{1}{1+u^2}du = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+u^2}du$$

$$= \frac{\pi}{2}\tan^{-1} u \Big|_{-1}^{1} = \frac{\pi}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{2}\right)^2.$$

＊Symmetry：

$f(x) = f(-x) \Leftrightarrow f$：even function (偶函數)⇔對稱 y 軸⇒$\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx.$

$f(x) = -f(x) \Leftrightarrow f$：odd function (奇函數)⇔對稱原點⇒$\int_{-a}^{a} f(x)dx = 0.$