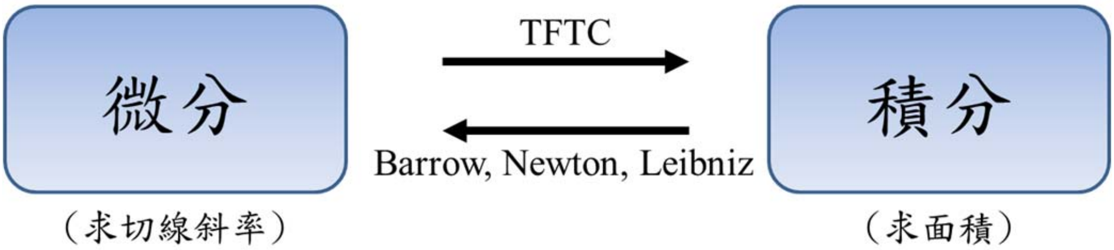
§5-3 The Fundamental Theorem of Calculus(TFTC)

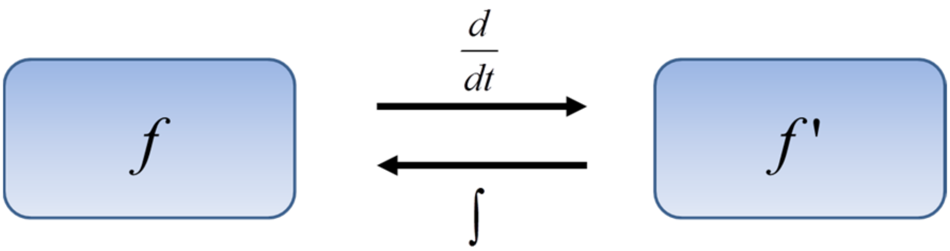


＊Theorem：(The fundamental theorem of calcucus)

Let f be continuous on [a , b].

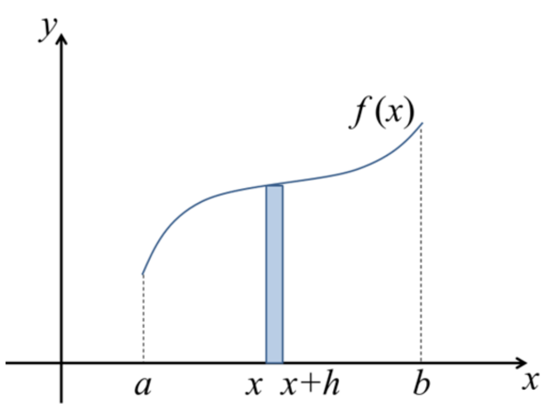
<u>Part I</u>：If $g(x)=\int_a^x f(t)dt \Rightarrow g'(x)=f(x)$.

<u>Part II</u>：$\int_a^b f(x)dx = F(x)|_a^b = F(b)-F(a)$, where F is any antiderivative of f.



＊Geometric Intuition

<u>Part I</u>

	$g(x+h)-g(x)\approx h\times f(x)$ $\Rightarrow \dfrac{g(x+h)-g(x)}{h}\approx f(x)$ $\Rightarrow g'(x)=f(x)$.

Part II

From part I $\Rightarrow g(x)$ is an antiderivative of f.

$$\Rightarrow F(x) = g(x) + c$$
$$\Rightarrow F(b) - F(a) = [g(b) + c] - [g(a) + c]$$
$$= g(b) - g(a) = \int_a^b f(t)dt$$

$$* \int_a^b f(t)dt = -\int_b^a f(t)dt$$

Example 1： $\int_1^4 \frac{1}{\sqrt{x}} dx = ?$

Solution：

$$\int_1^4 x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} |_1^4 = 4 - 2 = 2.$$

Example 2： $g(u) = \int_3^u \frac{1}{x^2 + x} dx$, find $\frac{dg}{du} = ?$

Solution：

$$\frac{dg}{du} = \frac{1}{u^2 + u}.$$

Example 3： $g(u) = \int_u^3 \frac{1}{x^2 + x} dx = -\int_3^u \frac{1}{x^2 + x} dx$, find $\frac{dg}{du} = ?$

Solution：

$$\frac{dg}{du} = -\frac{1}{u^2 + u}.$$

Example 4： $g(u)=\int_2^{\sqrt{u}}\frac{1}{x^2+x}dx$

$$=F\left(\sqrt{u}\right)-F(2)\text{, where } F'(x)=f(x)\text{, find } \frac{dg}{du}=?$$

Solution：

$$\frac{dg}{du}=F'\left(\sqrt{u}\right)\frac{1}{2}u^{-\frac{1}{2}}=f\left(\sqrt{u}\right)\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}\left(\frac{1}{u+\sqrt{u}}\right).$$

- $\frac{d}{dx}\int_a^{f(x)}g(t)dt=g\big(f(x)\big)f'(x).$
- $\frac{d}{dx}\int_{h(x)}^{f(x)}g(t)dt=g\big(f(x)\big)f'(x)-g\big(h(x)\big)h'(x).$

Example 5： $g(x)=\int_{\tan x}^{x^2}\frac{1}{\sqrt{2+t^4}}dt$, find $g'(x)$ = ?

Solution：

$$g'(x)=\frac{2x}{\sqrt{2+x^8}}-\frac{1}{\sqrt{2+\tan^4 x}}(\sec^2 x).$$

Example 6：If $F(x)=\int_1^x f(t)dt,\quad f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}du$, find $F''(2)=?$

Solution：

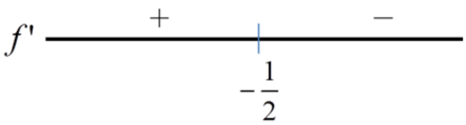
$$F'(x)=f(x)\Rightarrow F''(x)=f'(x)=\frac{\sqrt{1+x^8}(2x)}{x^2}$$

$$\Rightarrow F''(2)=\sqrt{1+2^8}=\sqrt{257}.$$

Example 7： Find the interval on which the curve $y = f(x) = \int_0^x \frac{1}{1+t+t^2}\,dt$ is concave upward.

Solution：

$$y' = \frac{1}{1+x+x^2} \Rightarrow y'' = \frac{-(1+2x)}{(1+x+x^2)^2}$$



$$\Rightarrow \left(-\infty, -\frac{1}{2}\right).$$

Example 8： $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = ?$

Solution：

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \int_0^1 \sqrt{x}\,dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}.$$

Example 9：Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \quad \text{for all } x > 0.$$

Solution：

(i) $6 + \int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \Rightarrow a = 9.$

(ii) $\frac{f(x)}{x^2} = x^{=\frac{1}{2}} \Rightarrow f(x) = x^{\frac{3}{2}}.$