§4-4 Indeterminate Form and L'Hospital's Rule

一、 Indeterminate form(不確定型)

$\frac{0}{0}$	$\frac{\infty}{\infty}=\infty\times 0$	1^{∞}
0^{0}	∞^{0}	$\infty-\infty$

＊註記：

The followings are determinate forms：

$\infty+\infty=\infty$

$\infty\times\infty=\infty$

二、 L'Hospital's Rule

1. $\frac{f}{g}$ ：indeterminate forms.

2. $\frac{f'}{g'}$ ：determinate forms.

$$\Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

＊註記：解題常遇的狀況

1. $\frac{f'}{g'}$ ：indeterminate form ⇒ same procedure can be continued.

   (See examples 2, 6, 8.)

2. $1^{\infty}, 0^{0}, \infty^{0}$ ⇒ take ln ⇒ $\frac{f}{g}$

   (See examples 4, 5.)

國立交通大學應用數學系 莊重教授

Example 1：

1. $\lim_{x\to 0}\frac{\sin x}{x}=?$ 2. $\lim_{x\to 1}\frac{x^2-1}{x-1}=?$

Solution：

1. $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1.$

2. $\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{2x}{1}=2.$

Example 2：$\lim_{x\to\infty}(x-\ln x)=?$

Solution：

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x(1-\frac{\ln x}{x})=\infty.\ (\because \lim_{x\to\infty}\frac{\ln x}{x}=0)$$

Example 3：$\lim_{x\to\infty}\frac{e^x}{x^2}=?$

Solution：

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty.$$

＊註記：

Generally speaking：

$x^x >> x! >> e^x >> x^n >> \ln x$ when x large.

Example 4： $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}=?$

Solution：

Let $y=(1+\sin 4x)^{\cot x}$

$\Rightarrow \ln y=\cot x\times\ln(1+\sin 4x)$

$=\dfrac{\ln(1+\sin 4x)}{\tan x}$

$\Rightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\dfrac{\ln(1+\sin 4x)}{\tan x}$

$=\lim_{x\to 0^+}\dfrac{4\cos 4x}{(\sec^2 x)\times(1+\sin 4x)}=4$.

$\Rightarrow \lim_{x\to 0^+}y=\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}=e^4$.

Example 5： $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=?$

Solution：

Let $y=\left(1+\dfrac{1}{x}\right)^x$

$\Rightarrow \ln y=x\ln\left(1+\dfrac{1}{x}\right)=\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$\Rightarrow \lim_{x\to\infty}\ln y=\lim_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\dfrac{1}{1+\frac{1}{x}}=1$

$$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Example 6： $\lim_{x\to 0}\frac{\tan x - x}{x^3} = ?$

Solution：

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec x\tan x}{6x} = \frac{1}{3}.$$

Example 7：

If f' is continuous, $f(2) = 0$, $f'(2) = 7$, evaluate

$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}.$$

Solution：

$$\begin{aligned}
&\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x} \\
&= \lim_{x\to 0}\frac{3f'(2+3x)+5f'(2+5x)}{1} \\
&= \lim_{x\to 0} 3f'(2)+5f'(2) = 56.
\end{aligned}$$

Example 8：For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

Solution：

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) \\
&= \lim_{x\to 0}\left(\frac{\sin 2x + ax^3 + bx}{x^3}\right) \\
&= \lim_{x\to 0}\left(\frac{2\cos 2x + 3ax^2 + b}{3x^2}\right)\left(\overset{\Delta}{=}\frac{f}{g}\right)
\end{aligned}$$

$\Rightarrow \dfrac{f}{g}$ must be indeterminate form

$\Rightarrow 2 + b = 0 \Rightarrow b = -2$

$$= \lim_{x \to 0} \left(\frac{2\cos 2x + 3ax^2 - 2}{3x^2} \right)$$

$$= \lim_{x \to 0} \left(\frac{-4\sin 2x + 6ax}{6x} \right)$$

\

$$= \lim_{x \to 0} \left(\frac{-8\cos 2x + 6a}{6} \right)$$

$$= \frac{1}{6}(-8 + 6a) = 0$$

$$\Rightarrow a = \frac{4}{3}.$$