§3-4 The Chain Rule

＊The chain rule：

I. Let $F(x) = f(g(x))$,

$\Rightarrow F'(x) = f'(g(x))g'(x)$.

II. Let $u = g(x)$, $y = F(x) = f(u)$,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

III. $y = a^x \Rightarrow y' = a^x \ln a$.

＊註記：

i. Key to apply the chain rule：Identufy ***u***.

ii. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$

Example 1：Let $y = (x^4 + 3x^2 - 2)^5$, find $y' = ?$

Solution：

$$y' = 5(x^4 + 3x^2 - 2)^4 \times (4x^3 + 6x).$$

Proof of (III)：

$$y = a^x = e^{\ln a^x} = e^{x \ln a}.$$

Let $u = x \ln a$,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \times \ln a = a^x \ln a.$$

Example 2：Let $y = e^{\sin x}$, find $y' = ?$

Solution：

i. Let $f(x) = e^x$, $g(x) = \sin x \Rightarrow f(g(x)) = e^{\sin x}$,

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$$\{f(g(x))\}' = f'(g(x))g'(x) = e^{\sin x} \times \cos x .$$

ii. Let $u = \sin x$,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^{u} \times \cos x = e^{\sin x} \cos x .$$

Example 3：Let $y = \sqrt{x + \sqrt{x}}$, find $y' = ?$

Solution：

Let $u = x + \sqrt{x}$,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}\left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2}(x + \sqrt{x})^{-\frac{1}{2}}\left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right).$$

Example 4：(Need to apply the chain rule three times)

Let $y = \tan\left(\sin e^{x^2}\right)$, find $y' = ?$

Solution：

Let $u = \sin e^{x^2}$, $v = e^{x^2}$, $w = x^2$,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dx} = \sec^2 u \times \cos v \times e^{w} \times 2x$$

$$= \sec^2\left(\sin e^{x^2}\right) \times \cos\left(e^{x^2}\right) \times e^{x^2} \times 2x .$$

Example 5：Let $y = \sqrt{\cot\left(e^{\sin x}\right)}$, find $y' = ?$

Solution：

$$y' = \frac{1}{2}\left(\cot e^{\sin x}\right)^{-\frac{1}{2}} \times \left[-\csc^2\left(e^{\sin x}\right)\right] \times e^{\sin x} \times \cos x .$$

註記：y 在 $x=0$ 的倒數不存在。

Example 6：Let $y = \sin[\cos(\tan x)]$, find $y' = ?$

Solution：

$$y' = \cos[\cos(\tan x)] \times [-\sin(\tan x)] \times \sec^2 x .$$

Example 7：Let $y = |x| = \sqrt{x^2}$, find $y' = ?$

Solution：

$$y' = \frac{1}{2}(x^2)^{-\frac{1}{2}} \times (2x) = \frac{x}{|x|}, \quad x \neq 0 .$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Example 8：Let $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution：

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} .$$

$$\frac{d^2y}{dx^2} = \left(\frac{d}{dx}\left(\frac{dy}{du}\right)\right) \times \frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2}$$
$$= \left(\frac{d^2y}{du^2}\right) \times \left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2} .$$

Proof of the chain rule：

Let $u = g(x)$, $y = f(u) = f(g(x))$.

Known that

$$\Delta u = g'(a)\Delta x - \varepsilon_1 \Delta x = [g'(a) - \varepsilon_1]\Delta x \quad \text{where} \quad \varepsilon_1 \to 0 \quad \text{as} \quad \Delta x \to 0 ,$$

and

$$\Delta y = f'(b)\Delta u - \varepsilon_2 \Delta u = [f'(b) - \varepsilon_2]\Delta u \quad \text{where} \quad \varepsilon_2 \to 0 \quad \text{as} \quad \Delta u \to 0 .$$

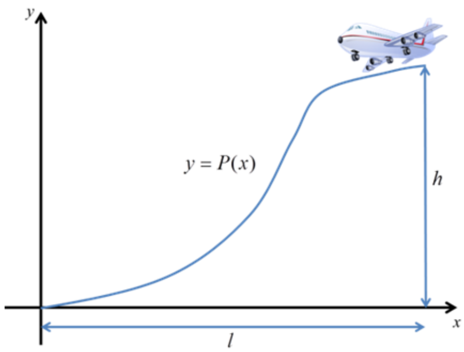
$$\Rightarrow \Delta y = [f'(b) - \varepsilon_2][g'(a) - \varepsilon_1]\Delta x$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = [f'(b) - \varepsilon_2][g'(a) - \varepsilon_1]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}[f'(b) - \varepsilon_2][g'(a) - \varepsilon_1] = f'(b)g'(a) = f'(g(a))g'(a) .$$

＊Applied project：Where should a pilot start descent?

An approach path for an aircraft landing is shown in the figure and satisfies the following conditions：



(i). The cruising altitude is h when descent starts at a horizontal distance l from touchdown at the origin.

(ii). The pilot must maintain a constant horizontal speed v throughout descent.

(iii). The absolute value of the vertical acceleration should not exceed a constant k (which is must less than the acceleration due to gravity).

1. Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.
2. Use conditions (ii) and (iii) to show that $\frac{6hv^2}{l^2} \le k$.
3. Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 1385\ km/h^2$. If the cruising altitude of a plane 11,000 m and the speed is $480\ km/h$, how far away from the airplane should the pilot start descent?
4. Graph the approach path if the conditions stated in Problem 3 are satisfied.

Solution：

1. Let $y(t)=f(t)=P(x(t))=ax^3+bx^2+cx+d$

$$\begin{cases} P(0)=0 \\ P'(0)=0 \\ P(l)=h \\ P'(l)=0 \end{cases} \Rightarrow \begin{cases} c=d=0 \\ al^3+bl^2=h \\ 3al^2+2bl=0 \end{cases} \Rightarrow \begin{cases} a=-\dfrac{2h}{l^3} \\ b=\dfrac{3h}{l^2} \\ c=0 \\ d=0 \end{cases}$$

$$\Rightarrow P(x)=-\frac{2h}{l^3}x^3+\frac{3h}{l^2}x^2$$

2. $$\frac{dy}{dt}=3ax^2\left(\frac{dx}{dt}\right)+2bx\left(\frac{dx}{dt}\right)$$

$$=3ax^2v+2bxv$$

$$\frac{d^2y}{dt^2}=6ax(t)v^2+2bv^2=-\frac{12h}{l^3}v^2x(t)+\frac{6h}{l^2}v^2$$

$$\Rightarrow\left|\frac{d^2y}{dt^2}\right|\le\frac{6h}{l^2}v^2 \quad \text{for} \quad 0\le x(t)\le l$$

$$\Rightarrow\frac{6h}{l^2}v^2\le k\,.$$

3. $$\frac{6\times(11)}{l^2}\times(480)^2\le 1385$$

$$\Rightarrow l\ge 104.78\,(km)\,.$$

4.

for $l = 104.78\,(km)$.