§3-1 Derivatives of Polynomials and Exponential Functions

＊這節將學多項式和指數的微分

公式：

i. $(x^n)' = nx^{n-1}$.

ii. $(e^x)' = e^x$.

iii. Let $a \in R$. Then $(af+g)' = af' + g'$ wherever f and g are differentiable.

Definition：

e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Proof of (i)：(For $n \in N$)

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

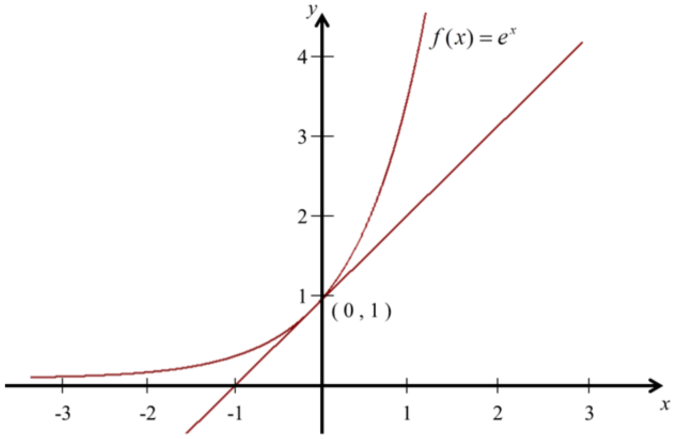
$$= \lim_{h \to 0} \frac{h(nx^{n-1} + C_2^n x^{n-2} h + C_3^n x^{n-3} h^2 + \ldots + C_{n-1}^n x h^{n-2} + h^{n-1})}{h}$$

$$= nx^{n-1}$$

＊註記：當 $n \in R$, (i)式也是對的，證明要用到 chain rule 和對數的微分。

Proof of (ii)

$$f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^{x}}{h}=\lim_{h\to 0}e^{x}\left(\frac{e^{h}-1}{h}\right)=e^{x}.$$



$$f'(0)=\lim_{h\to 0}\frac{e^{h}-1}{h}=1$$

Example 1：

$$(x^{3}+2x^{2}-3x+1)'=?$$

Solution：

$$(x^{3}+2x^{2}-3x+1)'=3x^{2}+4x-3.$$

Example 2：

Let $y=ae^{x}+\dfrac{b}{x}$, find $y'=?$

Solution：

$$y'=ae^{x}-\frac{b}{x^{2}}.$$

Example 3：Find the points on the curve $y = x^3 - x^2 - x + 1$, where the tangent is horizontal.

Solution：

Let $y' = 3x^2 - 2x - 1 = (x-1)(3x+1) = 0$, which means that there are two horizontal tangent lines at $x = 1$ & $x = -\frac{1}{3}$.

For $x = 1$ or $x = -\frac{1}{3}$, y = 0 or $y = \frac{32}{27}$, respectively.

⇒The points are (1 , 0) & $(-\frac{1}{3}, \frac{32}{27})$.

Example 4：

Find $y = ax^2 + bx$ whose tangent at (1 , 1) has equation $y = 3x - 2$.

Solution：

Passing (1 , 1) ⇒ $f'(1) = 3$

$$\Rightarrow \begin{cases} a + b = 1 \\ 2a + b = 3 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$$

Example 6：Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at (-2 , 6) & (2 , 0).

Solution：

$f'(x) = 3ax^2 + 2bx + c$

Passing (-2 , 6) ⇒ $-8a + 4b - 2c + d = 6$.

Passing (2 , 0) ⇒ $8a + 4b + 2c + d = 0$.

$f'(-2) = 0 \Rightarrow 12a - 4b + c = 0$

$f'(2) = 0 \Rightarrow 12a + 4b + c = 0$

$$\Rightarrow a = \frac{3}{16}, \quad b = 0, \quad c = -\frac{9}{4}, \quad d = 3.$$

Example 7：

$$\lim_{x \to 1} \frac{x^{2014} - 1}{x - 1} = ?$$

Solution：

$$\lim_{x \to 1} \frac{x^{2014} - 1}{x - 1} = f'(1), \text{ where } \quad f(x) = x^{2014}.$$

$$\Rightarrow f'(1) = 2014.$$

Example 8：

$$\text{Let } f(x) = \begin{cases} x^2 & , x \le 2 \\ mx + b & , x > 2 \end{cases}$$

Find the values of *m* and *b* that make *f* differentiable everywhere.

Solution：

$$f'(x) = \begin{cases} 2x & , x \le 2 \\ m & , x > 2 \end{cases}$$

Continuous at $x = 2 \Rightarrow \quad f(2^+) = f(2^-) \quad \Rightarrow \quad 4 = 2m + b$.

Differentiable at $x = 2 \Rightarrow \quad f'(2^+) = f'(2^-) \quad \Rightarrow \quad 4 = m$.

$$\Rightarrow b = -4.$$

Example 9：

Let $g(x)=\begin{cases}-1-2x, & x<-1\\ x^2, & -1\le x<1\\ x, & x\ge 1.\end{cases}$

i. Find the set of points at which g is continuous.

ii. Find the set of points at which g is differentiable.

Solution：

$$g'(x)=\begin{cases}-2, & x<-1\\ 2x, & -1\le x<1\\ 1, & x\ge 1.\end{cases}$$

i. $g(-1^-)=1=g(-1^+)$

$g(1^-)=1=g(1^+)$

The set of continuity of f is ***R***.

ii. $g'(-1^-)=-2=g'(-1^+)$

$g'(1^-)=2\neq 1=g'(1^+)$

The set of differentiability of f is ***R***－{1}.