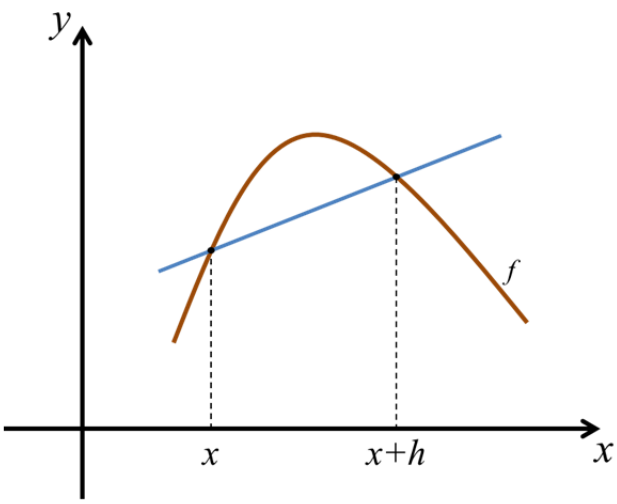
§2-8 The Derivative As a Function

Definition：

The derivative of $y = f(x)$ is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ whenever the limit exists.



Notations：

$*\ f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = Df(x) = D_x f(x)$.

$*\ f''(x) = y'' = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$

$*\ f'''(x) = y''' = \frac{d}{dx}(\frac{d^2y}{dx^2}) = \frac{d^3y}{dx^3}$

$*\ f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n}$

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Example 1： $f(x)=\sqrt{x-1}$. Compute $f'(x)$ by definition.

Solution：

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{(x+h)-1}-\sqrt{x-1}}{h}$$
$$=\lim_{h\to 0}\frac{[(x+h)-1]-(x-1)}{h(\sqrt{(x+h)-1}+\sqrt{x-1})}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{(x+h)-1}+\sqrt{x-1}}$$
$$=\frac{1}{2\sqrt{x-1}}$$

Example 2： $\lim_{x\to 0}\frac{3^x-1}{x}=f'(0)$. Find *f*.

Solution： $f(x)=3^x$

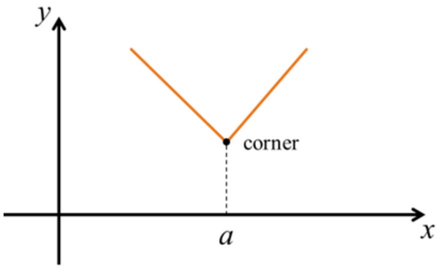
Theorem：*f* has derivative at $x = a$

$\Rightarrow$ *f* is continuous at $x = a$.

＊註記：反之不對！
見下頁之例子。

＊Some cases that yield the nonexistence of derivative of *f* at $x = a$：

(i)



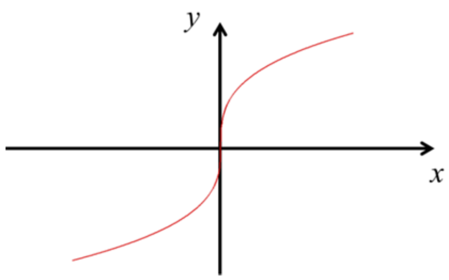
f is continuous at $x = a$,

but is not differentiable at $x = a$.

例如：$f(x)=|x|$.

f has no derivatives at $x=0$.

(ii)



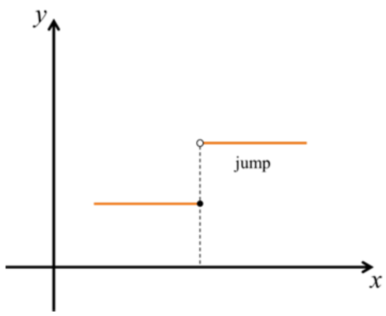
f has infinite tangent at $x = a$.

例如： $f(x) = x^{\frac{1}{3}}$.

f has no derivatives at x =0,

but f is continuous at x = 0.

(iii)



f has break at $x = a$.

例如：$f(x) = [x]$

f has no derivatives at all integers.

The set of discontinuous points

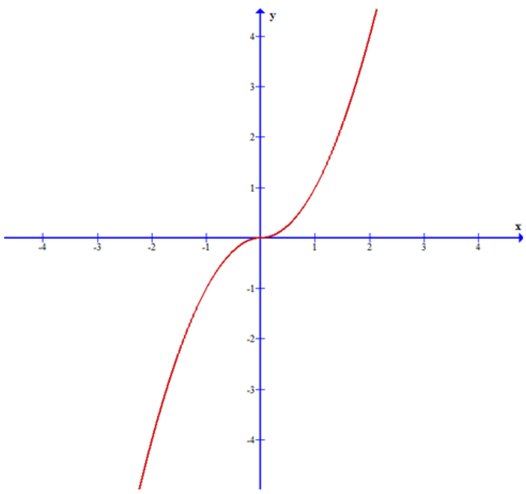
={all integers}.

∗Roughly speaking, if f has a break, a corner or a sharp turn at $x = a$, then f has no derivative there.

Example 3：$f(x) = x\,|\,x\,|$. Does f have a derivative at $x = 0$?

Solution：

$$f'(0) = \lim_{x\to 0}\frac{x\,|\,x\,|}{x} = \lim_{x\to 0}|\,x\,| = 0 .$$



Example 4： $f(x)=\begin{cases} x\sin\dfrac{1}{x} & ,if \quad x\neq 0 \\ 0 & ,if \quad x=0 \end{cases}$

Discuss the differentiability and continuity of f at $x = 0$.

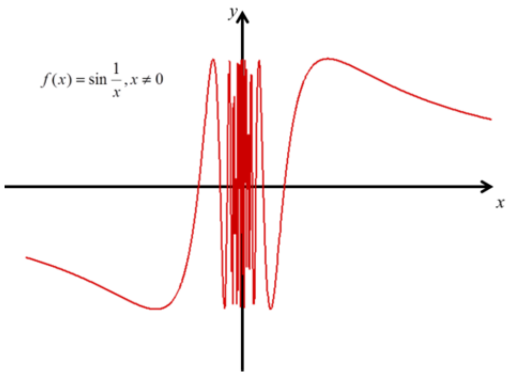
Solution：

i. f is continuous at $x = 0$ by Sandwich Theorem.

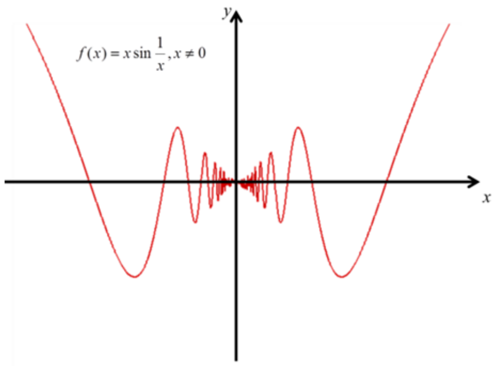
ii. $\lim\limits_{x\to 0}\dfrac{f(x)-0}{x-0}=\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ (not exist)

$\Rightarrow f$ is not differentiable at $x = 0$.

$\Rightarrow f$ is continuous at $x = 0$.



$f(x)=\sin\dfrac{1}{x}, x\neq 0$



$f(x)=x\sin\dfrac{1}{x}, x\neq 0$

Example 5： $f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & ,if \quad x\neq 0 \\ 0 & ,if \quad x=0 \end{cases}$

Discuss the differentiability and continuity of f at $x = 0$.

Solution：

$$f'(0)=\lim_{x\to 0}\frac{f(x)-0}{x-0}=\lim_{x\to 0}x\sin\frac{1}{x}=0.$$

$\Rightarrow f$ is differentiable at $x = 0$.

$\Rightarrow f$ is continuous at $x = 0$.

Example 6：Let $f(x)=\sqrt{ax+b}$, $f(0)$ = 1 and $f'(0)=1$. Find *a* and *b*.

Solution：

$$f(0)=1=\sqrt{b}\Rightarrow b=1\Rightarrow a=2.$$