Section 26

Characteristic Functions

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Characteristic function

Definition (characteristic function) The *characteristic function* of a random variable X is defined for real t by:

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} dF_X(x) = \int_{-\infty}^{\infty} \cos(tx) dF_X(x) + i \int_{-\infty}^{\infty} \sin(tx) dF_X(x).$$

- The characteristic function $\varphi(t) = M(it)$, where $M(t)$ is the moment generating function of random variable X.
- The characteristic function is the (inverse) Fourier transform of distribution function.
- The characteristic function always exist, because distribution function is always integrable.
- It is named the *characteristic function* since it completely *characterizes* the distribution.

Fundamental properties of characteristic functions

- The characteristic function of sum of two independent random variables is the product of individual characteristic functions.
- The characteristic function *uniquely* determines distribution function.
- From *the pointwise convergence* of characteristic function follows the *in-distribution convergence* of random variables.

Moments and characteristic function

- $\varphi(0) = 1$.
- $|\varphi(t)| \leq 1$ for all $t \in \Re$.
- $\varphi(t)$ is *uniformly continuous*.

Definition (uniform continuity) A function $f(t)$ is *uniformly continuous* in $\mathcal{A}$, if for any $\delta > 0$, there exists $h > 0$ such that

$$\sup_{\{(x,y)\in\mathcal{A}^2 \,:\, |x-y|\leq h\}} |f(x) - f(y)| < \delta.$$

Proof:

$$
\begin{aligned}
|\varphi(t+h)-\varphi(t)| &= \left|\int_{-\infty}^{\infty} e^{i(t+h)x}dF_X(x) - \int_{-\infty}^{\infty} e^{itx}dF_X(x)\right| \\
&= \left|\int_{-\infty}^{\infty} (e^{ihx}-1)e^{itx}dF_X(x)\right| \\
&\le \int_{-\infty}^{\infty} \left|(e^{ihx}-1)\right| \cdot \left|e^{itx}\right| dF_X(x) \\
&= \int_{-\infty}^{\infty} \left|(e^{ihx}-1)\right| dF_X(x), \\
\Bigg(&= \int_{-\infty}^{\infty} \sqrt{(\cos(hx)-1)^2+\sin^2(hx)}dF_X(x) \\
&= \int_{-\infty}^{\infty} \sqrt{2-2\cos(hx)}dF_X(x) \\
&= \int_{-\infty}^{\infty} 2\left|\sin(hx/2)\right| dF_X(x) \\
&= 2E\left[\left|\sin(hX/2)\right|\right] \Bigg)
\end{aligned}
$$

where the bound is independent of t. $\Box$.

Taylor expansion for characteristic functions

Theorem (Taylor's formula with remainder) Assume that $f(\cdot)$ has a continuous derivative of order $n+1$ in some open interval containing a. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

By the Taylor's formula with remainder about $x = 0$,

$$\begin{aligned} e^{ix} &= \sum_{k=0}^{n} \frac{(e^{ix})^{(k)}\big|_{x=0}}{k!}x^k + \frac{1}{n!}\int_0^x (x-t)^n (e^{it})^{(n+1)}dt \\ &= \sum_{k=0}^{n} \frac{i^k}{k!}x^k + \frac{1}{n!}\int_0^x (x-t)^n \left(i^{n+1}e^{it}\right) dt \\ &= \sum_{k=0}^{n} \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!}\int_0^x (x-t)^n e^{it}dt. \end{aligned}$$

Taylor expansion for characteristic functions

Integration by part for the below expression yields that:

$$\begin{aligned}\int_0^x (x-t)^{n-1} e^{it} dt &= -\frac{(x-t)^n}{n} e^{it}\Big|_0^x + \int_0^x \frac{(x-t)^n}{n} \left(ie^{it}\right) dt \\ &= \frac{x^n}{n} + \frac{i}{n}\int_0^x (x-t)^n e^{it} dt \\ &= \int_0^x (x-t)^{n-1} dt + \frac{i}{n}\int_0^x (x-t)^n e^{it} dt,\end{aligned}$$

which implies that:

$$\int_0^x (x-t)^{n-1}(e^{it}-1)dt = \frac{i}{n}\int_0^x (x-t)^n e^{it} dt,$$

This summarizes to:

$$e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} = \begin{cases} \dfrac{i^{n+1}}{n!}\displaystyle\int_0^x (x-t)^n e^{it} dt \\ \dfrac{i^n}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1}(e^{it}-1) dt \end{cases}$$

Taylor expansion for characteristic functions

Therefore, for $x \geq 0$,

$$\left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| = \begin{cases} \left|\frac{i^{n+1}}{n!}\int_0^x (x-t)^n e^{it} dt\right| \\ \left|\frac{i^n}{(n-1)!}\int_0^x (x-t)^{n-1}(e^{it}-1) dt\right| \end{cases}$$

$$\leq \begin{cases} \frac{1}{n!}\int_0^x |(x-t)^n| \cdot \left|e^{it}\right| dt \\ \frac{1}{(n-1)!}\int_0^x \left|(x-t)^{n-1}\right| \cdot \left|(e^{it}-1)\right| dt \end{cases}$$

$$\leq \begin{cases} \frac{1}{n!}\int_0^x (x-t)^n\, dt \\ \frac{2}{(n-1)!}\int_0^x (x-t)^{n-1}\, dt \end{cases}, \quad (\text{since } |e^{it}-1| \leq |e^{it}| + 1 = 2)$$

$$= \begin{cases} \frac{x^{n+1}}{(n+1)!} \\ \frac{2x^n}{n!} \end{cases}$$

and for $x < 0$,

$$\left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| = \begin{cases} \left|\frac{i^{n+1}}{n!}\int_0^x (x-t)^n e^{it} dt\right| \\ \left|\frac{i^n}{(n-1)!}\int_0^x (x-t)^{n-1}(e^{it}-1) dt\right| \end{cases}$$

$$\leq \begin{cases} \frac{1}{n!}\int_x^0 |(x-t)^n| \cdot \left|e^{it}\right| dt \\ \frac{1}{(n-1)!}\int_x^0 \left|(x-t)^{n-1}\right| \cdot \left|(e^{it}-1)\right| dt \end{cases}$$

$$\leq \begin{cases} \frac{1}{n!}\int_x^0 (t-x)^n \, dt \\ \frac{2}{(n-1)!}\int_x^0 (t-x)^{n-1} \, dt \end{cases}, \quad (\text{since } |e^{it}-1| \leq |e^{it}| + 1 = 2)$$

$$= \begin{cases} \frac{(-x)^{n+1}}{(n+1)!} \\ \frac{2(-x)^n}{n!} \end{cases}$$

Taylor expansion for characteristic functions

So to speak,

$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \le \min\left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}.$$

Lemma If X has a moment of order n, then

$$\left| \varphi(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!} E[X^k] \right| \le E\left[\min\left\{ \frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!} \right\} \right]$$

(if the moment of order $(n+1)$ also exists) $\le \min\left\{ \frac{|t|^{n+1} E[|X|^{n+1}]}{(n+1)!}, \frac{2|t|^n E[|X|^n]}{n!} \right\}.$

Proof:

$$\left| \varphi(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!} E[X^k] \right| = \left| \int_{\Re} e^{itx} dF_X(x) - \sum_{k=0}^{n} \frac{(it)^k}{k!} \int_{\Re} x^k dF_X(x) \right|$$

$$\le \int_{\Re} \left| e^{itx} - \sum_{k=0}^{n} \frac{(itx)^k}{k!} \right| dF_X(x).$$

□

Taylor expansion for characteristic functions

- **Concern:** When does the characteristic function have Taylor expansion?

Corollary If for any t,

$$\lim_{n\to\infty} \frac{|t|^n E[|X|^n]}{n!} = 0,$$

then

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k].$$

Theorem If

$$E[e^{|t||X|}] = \sum_{k=0}^{\infty} \frac{|t|^k}{k!} E[|X|^k] < \infty,$$

then

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k].$$

Taylor expansion for characteristic functions

Corollary If for some $t_0 \neq 0$,

$$\varphi(t_0) = \sum_{k=0}^{\infty} \frac{(it_0)^k}{k!} E[X^k],$$

then

1. for every $-|t_0| \leq t \leq |t_0|$,

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k].$$

2. $\varphi^{(k)}(0) = i^k E[X^k]$ for every $k = 0, 1, 2, \ldots$.

- Property 2, as a direct consequence of Property 1, is an analogous property to the moment generating function.
- However, moments can be determined by the characteristic function in a much weaker sense! (Notably, unlike the moment generation function which may have an unbounded jump at the origin, the characteristic function is uniformly continuous.)

Taylor expansion for characteristic functions

Lemma If $E[|X^k|] < \infty$, then

$$\varphi^{(k)}(0) = i^k E[X^k].$$

Proof:

1. $k = 1$**:**

$$\begin{aligned}
\left|\frac{\varphi(t+h)-\varphi(t)}{h} - E[iXe^{itX}]\right| &= \left|E\left[\frac{1}{h}e^{i(t+h)X} - \frac{1}{h}e^{itX} - iXe^{itX}\right]\right| \\
&= \left|E\left[e^{itX}\frac{(e^{ihX}-1-ihX)}{h}\right]\right| \\
&\le E\left[\left|e^{itX}\right| \cdot \left|\frac{e^{ihX}-1-ihX}{h}\right|\right] \\
&\le E\left[\frac{\min\{(1/2)|hX|^2, 2|hX|\}}{h}\right] \\
&= E\left[\min\{(1/2)h|X|^2, 2|X|\}\right].
\end{aligned}$$

By bounded convergence theorem, $i)$ $\min\{(1/2)h|X|^2, 2|X|\} \le 2|X|$, $ii)$ $|X|$ is integrable, and $iii)$ $\lim_{h\downarrow 0}\min\{(1/2)h|x|^2, 2|x|\} = 0$ almost everywhere jointly imply

$$\lim_{h\downarrow 0} E\left[\min\{(1/2)h|X|^2, 2|X|\}\right] = E\left[\lim_{h\downarrow 0}\min\{(1/2)h|X|^2, 2|X|\}\right] = 0.$$

Taylor expansion for characteristic functions

Theorem (bounded convergence theorem) If (i) $|f_n| \leq g$ almost everywhere, (ii) g is integrable, and (iii) $f_n \to f$ almost everywhere, then f is integrable and $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

Hence, we obtain:

$$\varphi'(t) = \lim_{h\downarrow 0} \frac{\varphi(t+h) - \varphi(t)}{h} = E[(iX)e^{itX}],$$

and

$$\varphi'(0) = E[iX].$$

2. $k = 2$**:**

$$
\begin{aligned}
\left|\frac{\varphi'(t+h)-\varphi'(t)}{h} - E[(iX)^2 e^{itX}]\right| &= \left|\frac{E[(iX)e^{i(t+h)X}] - E[(iX)e^{itX}]}{h} - E[(iX)^2 e^{itX}]\right| \\
&= \left|E\left[(iX)e^{itX}\frac{e^{ihX}-1-ihX}{h}\right]\right| \\
&\le E\left[|X|\cdot\left|e^{itX}\right|\cdot\left|\frac{e^{ihX}-1-ihX}{h}\right|\right] \\
&\le E\left[\min\{(1/2)h|X|^3, 2|X|^2\}\right],
\end{aligned}
$$

which again gives by bounded convergence theorem that

$$\varphi^{(2)}(t) = E[(iX)^2 e^{itX}],$$

and

$$\varphi^{(2)}(0) = E[(iX)^2].$$

3. Induction: The lemma can be proved by repeating the above process by induction (proving that the lemma holds for $k = n + 1$ under the premise that the lemma is valid for $k = n$). □

Taylor expansion for characteristic functions

- The above lemma does not imply that the derivatives of $\varphi(t)$ always exist. On the contrary, $\varphi(t)$ may be non-differentiable even if it is uniformly continuous.
- The above lemma actually tells us that the more moments X has, the more derivatives φ has.
- **Brainstorming:** Can this property be applied to Fourier Transform pair? Certainly, if the function is integrable.
- **Brainstorming:** In concept, the **tail** (rate-of-decaying) behavior of the distribution determines the **smoothness** (order of differentiability) of φ.

Riemann-Lebesgue theorem

Theorem 26.1 (Riemann-Lebesgue theorem) If X has a density, then $\varphi_X(t) \stackrel{|t|\to\infty}{\longrightarrow} 0$.

Every Lebesgue integrable function can be approximated by Riemann integrable functions of two kinds.

Theorem 17.1 Suppose that $\int_{\Re} |f(x)|dx < \infty$ (which indicates its Lebesgue integrability) and $\varepsilon > 0$ fixed.

1. There exists a step function $g(x) = \sum_{i=1}^{k} x_i I_{(a_i,b_i]}(x)$ (which indicates its Riemann integrability), where each a_i and b_i are finite real numbers, such that

$$\int_{\Re} |f(x) - g(x)|dx < \varepsilon.$$

2. There exists a continuous integrable h with bounded support (namely, which indicates its Riemann integrability) such that

$$\int_{\Re} |f(x) - h(x)|dx < \varepsilon.$$

Notably, a bounded measurable function is Riemann integrable on $[a, b]$ if it is continuous in $[a, b]$.

Proof: From Theorem 17.1, there is a step function $g(x) = \sum_{j=1}^{k} x_j I_{(a_j,b_j]}(x)$ such that

$$\int_{\Re} |f(x) - g(x)| dx < \varepsilon,$$

where $f(x)$ is the density of X, and satisfies $\int_{\Re} f(x)dx = 1 < \infty$. Therefore,

$$\begin{aligned}\left|\int_{\Re} f(x)e^{itx}dx - \int_{\Re} g(x)e^{itx}dx\right| &= \left|\int_{\Re} (f(x) - g(x))\, e^{itx}dx\right| \\ &\leq \int_{\Re} |f(x) - g(x)|\, dx < \varepsilon.\end{aligned}$$

The proof is completed by noting that ε is arbitrary and

$$\begin{aligned}\int_{\Re} g(x)e^{itx}dx &= \int_{\Re} \left(\sum_{j=1}^{k} x_j I_{(a_j,b_j]}(x)\right) e^{itx}dx \\ &= \sum_{j=1}^{k} x_j \int_{a_j}^{b_j} e^{itx}dx \\ &= \sum_{j=1}^{k} x_j \left(\frac{e^{itb_j} - e^{ita_j}}{it}\right) \stackrel{|t|\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

Independent and inversion

- The characteristic function of sum of independent random variables is the product of individual characteristic functions.
- The characteristic function of $-X$ is the complex conjugate of the characteristic function of X.

Inversion and uniqueness theorem

- How to prove that "The characteristic function uniquely determines the probability distribution."
- This is the task of **Uniqueness Theorem**.
- **Please again think of the counterpart theorem for Fourier transformation.**
- Also, remember that the cdf of a random variable is sufficient to define its statistic properties.

Inversion and uniqueness theorem

Example 18.4 Prove $S(T) = \int_0^T \frac{\sin(x)}{x}dx \overset{T\to\infty}{\longrightarrow} \frac{\pi}{2}$.

Proof:

$$\begin{aligned}
\int_0^T \frac{\sin(x)}{x}dx &= \int_0^T \sin(x)\left[\int_0^\infty e^{-ux}du\right]dx \\
&= \int_0^\infty \left[\int_0^T e^{-ux}\sin(x)dx\right]du \quad \text{(By Fubini's theorem)} \\
&= \int_0^\infty \frac{1}{1+u^2}\left[1 - e^{-uT}(u\sin(T)+\cos(T))\right]du \\
&= \int_0^\infty \frac{1}{1+u^2}du - \int_0^\infty \frac{e^{-uT}}{1+u^2}(u\sin(T)+\cos(T))du \\
&= \tan^{-1}(u)\Big|_0^\infty - \int_0^\infty \frac{e^{-uT}}{\sqrt{1+u^2}}\cos(T-\phi_u)du, \ \text{where} \ \cos(\phi_u) = 1/\sqrt{1+u^2} \\
&= \frac{\pi}{2} - \int_0^\infty \frac{e^{-uT}}{\sqrt{1+u^2}}\cos(T-\phi_u)du.
\end{aligned}$$

Finally,

$$\begin{aligned}\left|\int_0^\infty \frac{e^{-uT}}{\sqrt{1+u^2}}\cos(T-\phi_u)du\right| &\leq \int_0^\infty e^{-uT}\frac{1}{\sqrt{1+u^2}}\left|\cos(T-\phi_u)\right|du \\ &\leq \int_0^\infty e^{-uT}du, \text{ because } 1 \geq \frac{1}{\sqrt{1+u^2}} \text{ for } u \geq 0 \\ &= \frac{1}{T} \stackrel{T\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

Theorem (Fubini's theorem) Suppose $\mathcal{X}$ and $\mathcal{Y}$ can be expressed as $\mathcal{X} = \cup_{i=1}^\infty \mathcal{U}_i$ and $\mathcal{Y} = \cup_{i=1}^\infty \mathcal{V}_i$ for some $\{\mathcal{U}_i\}_{i=1}^\infty$ and $\{\mathcal{V}_i\}_{i=1}^\infty$ with $\mu(\mathcal{U}_i) < \infty$ and $\nu(\mathcal{V}_i) < \infty$. Then, if either

$$\int_{\mathcal{X}}\left(\int_{\mathcal{Y}}|f(x,y)|\nu(dy)\right)\mu(dx) < \infty$$

or

$$\int_{\mathcal{Y}}\left(\int_{\mathcal{X}}|f(x,y)|\mu(dx)\right)\nu(dy) < \infty,$$

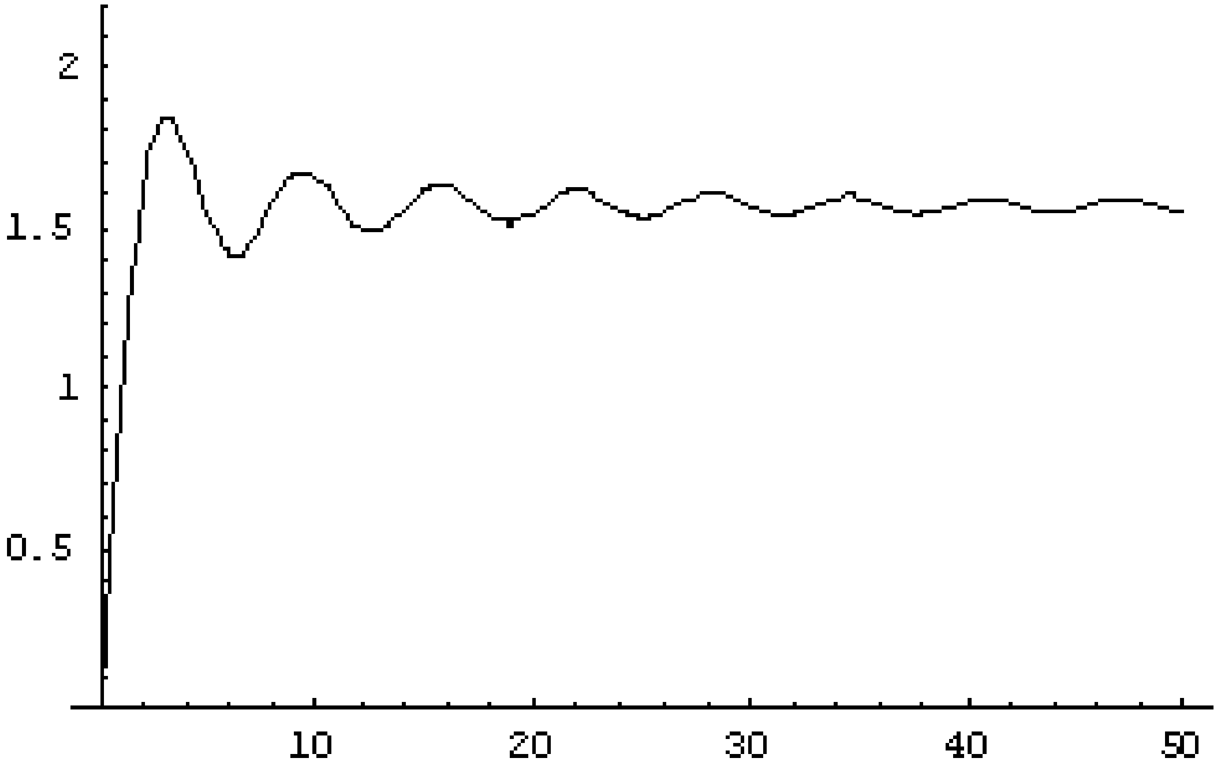
$$\int_{\mathcal{X}}\left(\int_{\mathcal{Y}}f(x,y)\nu(dy)\right)\mu(dx) = \int_{\mathcal{Y}}\left(\int_{\mathcal{X}}f(x,y)\mu(dx)\right)\nu(dy).$$

Checking:

$$\int_0^T \int_0^\infty \left|\sin(x)e^{-ux}\right| du\, dx = \int_0^T \frac{|\sin(x)|}{x} dx \leq S(\pi) + \frac{|T-\pi|}{\pi} < \infty.$$

Inversion and uniqueness theorem

$$\text{In fact}, 0 \leq S(T) \leq S(\pi).$$



Inversion and uniqueness theorem

Observation

$$\int_{-T}^{T} \frac{e^{itx}}{it} dt = 2\,\mathrm{sgn}(x) S(T|x|), \text{ where sgn } (x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Proof:

$$\begin{aligned} \int_{-T}^{T} \frac{e^{itx}}{it} dt &= \int_{-T}^{0} \frac{e^{itx}}{it} dt + \int_{0}^{T} \frac{e^{itx}}{it} dt \\ &= \int_{0}^{T} \frac{-e^{-itx}}{it} dt + \int_{0}^{T} \frac{e^{itx}}{it} dt \\ &= \int_{0}^{T} \frac{e^{itx} - e^{-itx}}{it} dt = 2 \int_{0}^{T} \frac{\sin(tx)}{t} dt \\ &= 2\,\mathrm{sgn}(x) \int_{0}^{T} \frac{\sin(t|x|)}{t} dt, \\ &= 2\,\mathrm{sgn}(x) \int_{0}^{T|x|} \frac{\sin(t')}{t'} dt' = 2\,\mathrm{sgn}(x) S(T|x|). \end{aligned}$$

□

- Actually, when $x = 0$, a rigorous statement should be $\int_{-T}^{T} \frac{1}{it} dt = \text{indeterminable}$.

Inversion and uniqueness theorem

Theorem 26.2 (uniqueness theorem) For any a and b with $\Pr[X = a] = \Pr[X = b] = 0$ and $a < b$,

$$\Pr[a < X \le b] = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)dt,$$

where $\varphi_X(t)$ is the characteristic function of random variable X.

Proof:

- Let I_T denote the quantity inside the limit, namely,

$$\begin{aligned} I_T &= \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)dt \\ &= \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\left(\int_{-\infty}^{\infty} e^{itx} dF_X(x)\right)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}dt\right)dF_X(x) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\big[\mathrm{sgn}(x-a)S(T|x-a|) - \mathrm{sgn}(x-b)S(T|x-b|)\big]dF_X(x). \end{aligned}$$

- The absolute value of the above integrand, namely $\big[\mathrm{sgn}(x-a)S(T|x-a|) - \mathrm{sgn}(x-b)S(T|x-b|)\big]$, is bounded above by $2S(\pi)$, which is integrable with respect to measure $dF_X(x)$.

Hence, by bounded convergence theorem,

$$\begin{aligned}\lim_{T\to\infty} I_T &= \frac{1}{\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \left[\operatorname{sgn}(x-a)S(T|x-a|) - \operatorname{sgn}(x-b)S(T|x-b|)\right] dF_X(x)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \psi_{a,b}(x)dF_X(x),\end{aligned}$$

where

$$\psi_{a,b}(x) = \lim_{T\to\infty} [\text{sgn}(x-a)S(T|x-a|) - \text{sgn}(x-b)S(T|x-b|)]$$

$$= \lim_{T\to\infty} \begin{cases} S(T|x-b|) - S(T|x-a|), & \text{for } x < a; \\ S(T|x-b|), & \text{for } x = a; \quad \text{indeterminable!} \\ S(T|x-a|) + S(T|x-b|), & \text{for } a < x < b; \\ S(T|x-a|), & \text{for } x = b; \quad \text{indeterminable!} \\ S(T|x-a|) - S(T|x-b|), & \text{for } x > b \end{cases}$$

$$= \begin{cases} 0, & \text{for } x < a; \\ \frac{\pi}{2}, & \text{for } x = a; \quad \text{indeterminable!} \\ \pi, & \text{for } a < x < b; \\ \frac{\pi}{2}, & \text{for } x = b; \quad \text{indeterminable!} \\ 0, & \text{for } x > b. \end{cases}$$

Consequently,

$$\lim_{T\to\infty} I_T = \frac{1}{\pi}\int_{-\infty}^{\infty} \psi_{a,b}(x)dF_X(x)$$

$$= \frac{1}{2}\Pr[X = a] + \Pr[a < X < b] + \frac{1}{2}\Pr[X = b].$$

The proof is completed by noting that $\Pr[X = a] = \Pr[X = b] = 0$. $\square$

Remarks:

- We can tell more from the proof (than from the statement of the theorem).
- For example,

Theorem If $\int_{-\infty}^{\infty} |\varphi_X(t)| dt < \infty$, then no point mass exists.
I.e., $\Pr[X = a] = 0$ for every $a \in \Re$.

Proof:

Inversion and uniqueness theorem

$$
\begin{aligned}
|e^{ix} - 1| &\le \min\left\{\frac{1}{1!}|x|, \frac{1}{0!}2|x|^0\right\} \\
|e^{ix} - (1 + ix)| &\le \min\left\{\frac{1}{2!}|x|^2, \frac{1}{1!}2|x|\right\} \\
\left|e^{ix} - \left(1 + ix + \frac{1}{2}(ix)^2\right)\right| &\le \min\left\{\frac{1}{3!}|x|^3, \frac{1}{2!}2|x|^2\right\} \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\Pr[X = a] + \Pr[a < X < b] + \frac{1}{2}\Pr[X = b] &= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)dt \\
&\le \limsup_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \left|\frac{e^{-ita} - e^{-itb}}{it}\right| |\varphi_X(t)|\, dt \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{e^{it(b-a)} - 1}{t(b-a)}\right| |b-a|\, |\varphi_X(t)|\, dt \\
&\le \frac{|b-a|}{2\pi}\int_{-\infty}^{\infty} |\varphi_X(t)|\, dt.
\end{aligned}
$$

Thus, when taking $a = b$, we immediately obtain $\Pr[X = a] \le 0$. $\square$

Inversion and uniqueness theorem

- Another example of "We can tell more from the proof." is:

Theorem If $\int_{-\infty}^{\infty} |\varphi_X(t)| dt < \infty$, then the random variable X has density f, and its density (can be made to) satisfies:

1. $f_X(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \varphi_X(t) e^{-itx} dt$;
2. $f_X(x) = F'_X(x)$ for every $x \in \Re$;
3. $f_X(x)$ is uniformly continuous for $x \in \Re$.

Proof: First, from Slide 26-28, we know that random variable X has no point mass. Then, by Theorem 26.2, we have that for every x (as $\Pr[X = x] = 0$ and $\Pr[X = x + h] = 0$)

$$\frac{F_X(x+h) - F_X(x)}{h} = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} I_{[-T,T]} \frac{e^{-itx} - e^{-it(x+h)}}{ith} \varphi_X(t) dt.$$

As

$$\left| I_{[-T,T]} \frac{e^{-itx} - e^{-it(x+h)}}{ith} \varphi_X(t) \right| \leq \left| \frac{e^{ith} - 1}{th} \right| |\varphi_X(t)| \leq |\varphi_X(t)|,$$

which is integrable with respect to Lebesgue measure, dominated convergence theorem implies that

$$\frac{F_X(x+h) - F_X(x)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-it(x+h)}}{ith}\varphi_X(t)dt.$$

Using dominated convergence theorem again with respect to h yields:

$$\begin{aligned}
\lim_{h\to 0}\frac{F_X(x+h) - F_X(x)}{h} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{h\to 0}\frac{e^{-itx} - e^{-it(x+h)}}{ith}\varphi_X(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t)\frac{-1}{it}\left(\lim_{h\to 0}\frac{e^{-it(x+h)} - e^{-itx}}{h}\right) dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t)e^{-itx}dt.
\end{aligned}$$

We can similarly show that:

$$\lim_{h\to 0}\frac{F_X(x) - F_X(x-h)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t)e^{-itx}dt.$$

Accordingly,

$$F_X'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t)e^{-itx}dt.$$

Inversion and uniqueness theorem

- *Claim:* $F'_X(x)$ is uniformly continuous in $x \in \Re$.

  Proof:

$$
\begin{aligned}
|F'_X(x+h) - F'_X(x)| &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \varphi_X(t) e^{it(x+h)} dt - \int_{-\infty}^{\infty} \varphi_X(t) e^{itx} dt\right| \\
&= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \varphi_X(t) e^{itx}(e^{iht} - 1) dt\right| \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\varphi_X(t)| \left|e^{itx}\right| \left|e^{iht} - 1\right| dt \\
&= \frac{\int_{-\infty}^{\infty} |\varphi_X(t)| dt}{2\pi} \int_{-\infty}^{\infty} \left|e^{iht} - 1\right| f_U(t) dt \\
&\qquad \text{where } f_U(t) = |\varphi_X(t)| \bigg/ \int_{-\infty}^{\infty} |\varphi_X(t)| dt \\
&= \frac{1}{\pi}\left(\int_{-\infty}^{\infty} |\varphi_X(t)| dt\right) E[|\sin(hU/2)|],
\end{aligned}
$$

where the upper bound is independent of x. ■

Inversion and uniqueness theorem

- The proof is then completed by quoting the following theorem. In other words, since F' satisfies that

$$\int_a^b F'(x)dx = F(b) - F(a)$$

for every $a, b \in \Re$, it is surely a density of X. $\Box$

Theorem (See Page 224 in Section 17) If F is a function with continuous derivative F', then

$$\int_a^b F'(x)dx = F(b) - F(a).$$

- Note that the density is not unique. Any function g satisfying

$$\int_a^b g(x)dx = F(b) - F(a)$$

for every $a, b \in \Re$ is a density of X.

Inversion and uniqueness theorem

Summary of properties of cdf F

- F is differentiable almost everywhere.

  Theorem (H. L. Royden, *Real Analysis*, 3rd edition, pp. 100, 1988) Let g be an increasing real-valued function on the interval $[a, b]$. Then, g is differentiable almost everywhere. The derivative g' is measurable, and

  $$\int_a^b g'(x)dx \le g(b) - g(a).$$

  By this theorem, together with the fact that the number of intervals that F is increasing is countable, cdf F is almost everywhere differentiable.

- F is not necessarily differentiable on every x.

- F may not have density.

Inversion and uniqueness theorem

- If F has density f, then $F' = f$ almost everywhere.
- If F has continuous density f, then $F' = f$ everywhere.

  By definition of density,

  $$\frac{1}{h}\int_{x}^{x+h} f(x)dx = \frac{F(x+h) - F(x)}{h}.$$

  This can be used to show the above two statements.
- If F' is continuous, then F' (can be made to) be the density of F.

Remarks

- The **tail** behavior of the characteristic function determines the **smoothness** of f_X.
- The **tail** behavior of the density (i.e., f_X) determines the **smoothness** of φ_X.

Characteristic functions of some known distributions

- **Standard normal:** $\begin{cases} f_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ for } -\infty < x < \infty. \\ \varphi_X(t) = e^{-t^2/2}. \end{cases}$

  First,

$$\int_{-\infty}^{\infty} |\varphi_X(t)| dt = \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi} < \infty.$$

  Hence, $f_X(x)$ can be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t) e^{-itx} dt$, and is also uniformly continuous.

  As $\varphi_X(t)$ decays exponentially fast at $|t| \to \infty$, $f_X(x)$ has all orders of derivative and is very, very smooth in its shape.

Characteristic functions of some known distributions

Furthermore,

$$
\begin{aligned}
\frac{|t|^n E[|X|^n]}{n!} &= \frac{|t|^n \frac{2^{n/2}\Gamma((n+1)/2)}{\sqrt{\pi}}}{n!} \\
&= \begin{cases} \dfrac{|t|^{2j-1} 2^j (j-1)!}{(2j-1)!\sqrt{2\pi}} & \text{for } n = 2j-1 \text{ odd;} \\[2ex] \dfrac{|t|^{2j} 2^j \Gamma((2j+1)/2)}{(2j)!\sqrt{\pi}} & \text{for } n = 2j \text{ even} \end{cases} \\
&\le \max\left\{\frac{1}{|t|}, 1\right\} \frac{(2|t|^2)^j}{j!} \text{ respectively for } n = 2j-1 \text{ odd and } n = 2j \text{ even} \\
&\le \max\left\{\frac{1}{|t|}, 1\right\} \left(\frac{2e|t|^2}{j}\right)^j \quad \text{since } j! > (j/e)^j \\
&\xrightarrow{n\to\infty} 0
\end{aligned}
$$

Stirling's approximation

$$
\sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n-1}\right).
$$

Gamma function $\Gamma(x+1) = x \cdot \Gamma(x)$ for $x > 0$ and $\Gamma(1/2) = \sqrt{\pi}$.

implies that

$$\begin{aligned}
\varphi_X(t) &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k] \\
&= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left(\frac{2^{(k-2)/2}(1 + (-1)^k)\Gamma((k+1)/2)}{\sqrt{\pi}} \right) \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left(\frac{2^j \Gamma((2j+1)/2)}{\sqrt{\pi}} \right) t^{2j} \qquad (\text{Take } k = 2j) \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{t^2}{2} \right)^j \quad \left(= e^{-t^2/2} \right).
\end{aligned}$$

Characteristic functions of some known distributions

- **Uniform:** $\begin{cases} f_X(x) & = & 1 \text{ for } 0 < x < 1. \\ \varphi_X(t) & = & \dfrac{e^{it}-1}{it}. \end{cases}$

First,

$$\int_{-\infty}^{\infty} |\varphi_X(t)| dt = \int_{-\infty}^{\infty} \frac{|\sin(t/2)|}{|t/2|} dt = 4\int_0^{\infty} \frac{|\sin(s)|}{s} ds = \infty.$$

Hence, $f_X(x)$ cannot be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t) e^{-itx} dt$, and is not uniformly continuous as it has jumps at $x = 0$ and $x = 1$.

In addition,

$$\lim_{n\to\infty} \frac{|t|^n E[|X|^n]}{n!} = \lim_{n\to\infty} \frac{|t|^n (1/(n+1))}{n!} = \lim_{n\to\infty} \frac{|t|^n}{(n+1)!} = 0$$

implies that

$$\begin{aligned} \varphi_X(t) &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k] \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left(\frac{1}{k+1}\right) \\ &= \frac{1}{it} \sum_{k=0}^{\infty} \frac{(it)^{k+1}}{(k+1)!} \quad \left(= \frac{e^{it}-1}{it}\right). \end{aligned}$$

- **Exponential:** $\begin{cases} f_X(x) &= e^{-x} \text{ for } 0 < x < \infty. \\ \varphi_X(t) &= \dfrac{1}{1-it}. \end{cases}$

First,

$$\int_{-\infty}^{\infty} \left|\frac{1}{1-it}\right| dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+t^2}} dt = 2\int_0^{\infty} \frac{1}{\sqrt{1+t^2}} dt = \infty.$$

Hence, $f_X(x)$ cannot be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t) e^{-itx} dt$, and is not uniformly continuous as it has a jump at $x = 0$.

In addition, for $|t| < 1$,

$$\lim_{n\to\infty} \frac{|t|^n E[|X|^n]}{n!} = \lim_{n\to\infty} \frac{|t|^n n!}{n!} = \lim_{n\to\infty} |t|^n = 0$$

$$\left(E[e^{|tX|}] = \int_0^{\infty} e^{|t|x} e^{-x} dx = \int_0^{\infty} e^{-(1-|t|)x} dx = \frac{1}{1-|t|} < \infty \right)$$

implies that for $|t| < 1$,

$$\varphi_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k] = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} (k!) = \sum_{k=0}^{\infty} (it)^k = \frac{1}{1-it}.$$

Characteristic functions of some known distributions

- **Double exponential (Bilateral exponential):** $\begin{cases} f_X(x) = \frac{1}{2}e^{-|x|} \text{ for } -\infty < x < \infty. \\ \varphi_X(t) = \frac{1}{1+t^2}. \end{cases}$

  First,

$$\int_{-\infty}^{\infty} \left| \frac{1}{1+t^2} \right| dt = \pi < \infty.$$

  Hence, $f_X(x)$ can be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(t)e^{-itx}dt$, and also is uniformly continuous.
  But it is not differentiable at $x = 0$. (Check "tail" of $\varphi(t)$!)
  In addition, for $|t| < 1$,

$$\lim_{n\to\infty} \frac{|t|^n E[|X|^n]}{n!} = \lim_{n\to\infty} \frac{|t|^n (n!)}{n!} = \lim_{n\to\infty} |t|^n = 0$$

$$\left(E[e^{|tX|}] = \int_{-\infty}^{\infty} e^{|t||x|}\frac{1}{2}e^{-|x|}dx = \int_0^{\infty} e^{-(1-|t|)x}dx = \frac{1}{1-|t|} < \infty \right)$$

  implies that for $|t| < 1$,

$$\varphi_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}E[X^k] = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}\left(\frac{1}{2}(1+(-1)^k)k!\right) = \sum_{j=0}^{\infty}(it)^{2j} = \frac{1}{1+t^2}.$$

Characteristic functions of some known distributions

- **Cauchy:** $\begin{cases} f_X(x) = \dfrac{1}{\pi(1+x^2)} \text{ for } -\infty < x < \infty. \\ \varphi_X(t) = e^{-|t|}. \end{cases}$

  First,

$$\int_{-\infty}^{\infty} \left| e^{-|t|} \right| dt = 2 < \infty.$$

Hence, $f_X(x)$ can be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(t)e^{-itx}dt$, and is uniformly continuous and differentiable (for $\varphi_X(t)$ decays very fast at $|t| \to \infty$ and hence $f_X(x)$ is very smooth).

In addition,

$$\lim_{n\to\infty} \frac{|t|^n E[|X|^n]}{n!} = \begin{cases} \infty, & \text{for } |t| > 0; \\ \text{indeterminate}, & \text{for } |t| = 0, \text{ (due to the indeterminate of } 0 \times \infty) \end{cases}$$

and $\varphi_X(t)$ exhibits no Taylor-expansion expression (at the origin).

Characteristic functions of some known distributions

- **Triangular:** $\begin{cases} f_X(x) = 1 - |x| \text{ for } -1 < x < 1. \\ \varphi_X(t) = 2\left(\dfrac{1-\cos(t)}{t^2}\right). \end{cases}$

  Notably,

  $$\int_{-\infty}^{\infty} 2\left|\frac{1-\cos(t)}{t^2}\right| dt = 2\pi < \infty.$$

  Hence, $f_X(x)$ can be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(t)e^{-itx}dt$, and also is uniformly continuous.
  But it is not differentiable at $x = -1, 0, 1$. (Check "tail" of $\varphi(t)$!)
  In addition,

  $$\lim_{n\to\infty}\frac{|t|^n E[|X|^n]}{n!} = \lim_{n\to\infty}\frac{|t|^n(2/[(n+2)(n+1)])}{n!} = \lim_{n\to\infty}\frac{2|t|^n}{(n+2)!} = 0$$

  implies that

  $$\varphi_X(t) = \sum_{k=0}^{\infty}\frac{(it)^k}{k!}E[X^k] = \sum_{k=0}^{\infty}\frac{(it)^k}{k!}\left(\frac{1+(-1)^k}{(k+1)(k+2)}\right) = 2\sum_{j=0}^{\infty}\frac{(-)^j(t^2)^j}{(2j+2)!} = 2\frac{1-\cos(t)}{t^2}.$$

Characteristic functions of some known distributions

- ***No name***: $\begin{cases} f_X(x) &= \dfrac{1}{\pi}\left(\dfrac{1-\cos(x)}{x^2}\right) \text{ for } -\infty < x < \infty. \\ \varphi_X(t) &= (1-|t|)I_{(-1,1)}(t). \end{cases}$

  Notably,

$$\int_{-\infty}^{\infty} \left|(1-|t|)I_{(-1,1)}(t)\right| dt = 1 < \infty.$$

  Hence, $f_X(x)$ can be recovered from $\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(t)e^{-itx}dt$, and also is uniformly continuous and differentiable (for $\varphi_X(t)$ decays very, very fast at $|t| \to \infty$, and hence $f_X(x)$ has all orders of derivatives and is very, very smooth).
  In addition, $E[|X|^n] = \infty$ for $n \geq 3$; hence,

$$\lim_{n\to\infty} \frac{|t|^n E[|X|^n]}{n!} = \begin{cases} \infty, & \text{for } |t| > 0; \\ \text{indeterminate}, & \text{for } |t| = 0, \end{cases}$$

  and $\varphi_X(t)$ exhibits no Taylor-expansion expression.

Continuity theorem

Theorem 26.3 (Continuity theorem)

$$X_n \Rightarrow X \text{ if, and only if, } \varphi_{X_n}(t) \to \varphi_X(t) \text{ for every } t \in \Re.$$

Proof:

1. $X_n \Rightarrow X$ **implies** $\varphi_{X_n}(t) \stackrel{n\to\infty}{\longrightarrow} \varphi_X(t)$.

Theorem 25.8 (A rephrased version) The following two conditions are equivalent.

- $F_n \Rightarrow F$;
- $\lim_{n\to\infty} \int_{\Re} f(x) dF_n(x) = \int_{\Re} f(x) dF(x)$ for every bounded, continuous real function f.

From Theorem 25.8, $X_n \Rightarrow X$ implies that

$$\lim_{n\to\infty} \int_{\Re} \cos(tx) dF_{X_n}(x) = \int_{\Re} \cos(tx) dF_X(x)$$

and

$$\lim_{n\to\infty} \int_{\Re} \sin(tx) dF_{X_n}(x) = \int_{\Re} \sin(tx) dF_X(x).$$

These two jointly give that $\varphi_{X_n}(t) \stackrel{n\to\infty}{\longrightarrow} \varphi_X(t)$.

2. $\varphi_{X_n}(t) \stackrel{n\to\infty}{\longrightarrow} \varphi_X(t)$ **implies** $X_n \Rightarrow X$.

We will first show that $\varphi_{X_n}(t) \stackrel{n\to\infty}{\longrightarrow} \varphi_X(t)$ implies $\{X_n\}_{n\geq 1}$ (or equivalently, $\{F_n\}_{n\geq 1}$) is tight. Then, by Helly's Theorem, there exists a subsequence $\{X_{n_k}\}_{k\geq 1}$ such that $X_{n_k} \Rightarrow Y$, where Y is a random variable with legitimate cdf $F_Y(\cdot)$.

Theorem 25.9 (Helly's theorem) For every sequence $\{F_n\}_{n=1}^{\infty}$ of distribution functions, there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ and a non-decreasing, right-continuous function F (not necessarily a cdf) such that

$$\lim_{k\to\infty} F_{n_k}(x) = F(x)$$

for every continuous points of F.

Theorem 25.10 (rephrased version) Tightness of $\{F_{n_k}\}_{k=1}^{\infty}$ is a necessary and sufficient condition for the limit $F(\cdot)$ in Helly's theorem to be a cdf.

Definition (tightness) A sequence of cdf's is said to be *tight* if for any $\varepsilon > 0$, there exist x and y such that

$$F_n(x) < \varepsilon \text{ and } F_n(y) > 1 - \varepsilon \text{ for all sufficiently large } n.$$

Continuity theorem

By Fubini's theorem,

$$\begin{aligned}
\frac{1}{u}\int_{-u}^{u}(1-\varphi_{X_n}(t))\,dt &= \frac{1}{u}\int_{-u}^{u}\left(\int_{-\infty}^{\infty}(1-e^{itx})dF_n(x)\right)dt \\
&= \frac{1}{u}\int_{-\infty}^{\infty}\left(\int_{-u}^{u}(1-e^{itx})dt\right)dF_n(x) \\
&= 2\int_{-\infty}^{\infty}\left(1-\frac{\sin(ux)}{ux}\right)dF_n(x)
\end{aligned}$$

(This shows that $\frac{1}{u}\int_{-u}^{u}(1-\varphi_{X_n}(t))\,dt$ is a non-negative real number.)

$$\begin{aligned}
&\geq 2\int_{[|x|\geq 2/u]}\left(1-\frac{\sin(ux)}{ux}\right)dF_n(x) \quad \left(\text{since } 1-\frac{\sin(ux)}{ux}\geq 0\right) \\
&\geq 2\int_{[|x|\geq 2/u]}\left(1-\frac{1}{|ux|}\right)dF_n(x) \\
&\geq \int_{[|x|\geq 2/u]}dF_n(x) \\
&= \Pr\left[|X_n|\geq\frac{2}{u}\right].
\end{aligned}$$

Continuity theorem

Checking for Fubini's theorem:

$$\int_{-u}^{u}\int_{-\infty}^{\infty}|1-e^{itx}|\,dF_n(x)\,dt \le \int_{-u}^{u} 2\,dt = 4u < \infty.$$

With the above inequality, we next prove that the sequence of probability measures $\{F_{X_n}\}_{n=1}^{\infty}$ is tight.

Fix an $\varepsilon > 0$. Since $\varphi_X(t)$ is (uniformly) continuous and $\varphi_X(0) = 1$, there exists u (small enough) such that

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_X(t))dt < \frac{\varepsilon}{2}.$$

Continuity at $t = 0$ means that for $\varepsilon/4 > 0$, there exists $u > 0$ such that $|1-\varphi_X(t)| < \varepsilon/4$ for $-u \le t \le u$.

By bounded convergence theorem,

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_{X_n}(t))dt \stackrel{n\to\infty}{\longrightarrow} \frac{1}{u}\int_{-u}^{u}(1-\varphi_X(t))dt.$$

Continuity theorem

Hence, there exists N_0 such that for $n > N_0$,

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_{X_n}(t))dt < \varepsilon,$$

which by the previously obtained inequality confirms that

$$\Pr\left[|X_n| > \frac{2}{u}\right] < \varepsilon \text{ for all } n \geq N_0,$$

namely, $\{F_{X_n}\}_{n=1}^{\infty}$ is tight.

Now suppose that $X_{n_k} \Rightarrow Y$ for some Y. Then, by the first part of the proof, we obtain that for every t,

$$\varphi_{X_{n_k}}(t) \stackrel{k\to\infty}{\longrightarrow} \varphi_Y(t).$$

However, we are given that for every t,

$$\varphi_{X_{n_k}}(t) \stackrel{k\to\infty}{\longrightarrow} \varphi_X(t).$$

These two limits should coincide in every t, i.e., $\varphi_X(t) = \varphi_Y(t)$.

We can then infer by the Uniqueness Theorem that $X_{n_k} \Rightarrow X$.

Theorem 26.2 (uniqueness theorem) For any a and b with $\Pr[X = a] = \Pr[X = b] = 0$ and $a < b$,

$$\Pr[a < X \le b] = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)dt,$$

where $\varphi_X(t)$ is the characteristic function of random variable X.

In fact, we know by $\varphi_{X_n}(t) \stackrel{n\to\infty}{\longrightarrow} \varphi_X(t)$ and the Uniqueness Theorem that for every subsequence $\{n_j\}_{j\ge1}$ satisfying that $\{X_{n_j}\}_{j\ge1}$ converges in distribution to some limiting random variable, this limiting random variable should be X.

Finally, we will prove $X_n \Rightarrow X$ by contradiction.

Suppose X_n does not converge in distribution to X. Then, there exists x with $\Pr[X = x] = 0$ such that

$$\Pr[X_n \le x] \not\to \Pr[X \le x],$$

which implies the existence of subsequence $\{n_j\}_{j\ge1}$ such that

$$\liminf_{j\to\infty} \left|\Pr[X_{n_j} \le x] - \Pr[X \le x]\right| > 0.$$

Continuity theorem

However, for this $\{X_{n_j}\}_{j\geq 1}$, we can find a further subsequence such that $X_{n_{j_k}} \Rightarrow X$. This contradicts to

$$\liminf_{j\to\infty} \left|\Pr[X_{n_j} \leq x] - \Pr[X \leq x]\right| > 0.$$

We therefore obtain the desired contradiction. $\square$

Continuity theorem

- The next two corollaries can be proved similarly using the above proof.
- But their statements could be more useful in applications.

Corollary 1 Suppose a sequence of characteristic functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ has limits in every t, namely $\lim_{n\to\infty} \varphi_n(t)$ exists for every t.
Define

$$g(t) = \lim_{n\to\infty} \varphi_n(t).$$

Then if $g(t)$ is continuous at $t = 0$, then there exists a probability measure μ such that

$$\mu_n \Rightarrow \mu, \text{ and } \mu \text{ has characteristic function } g$$

where μ_n is the probability measure corresponding to characteristic function $\varphi_n(\cdot)$.

Proof: Notably, in the second part of the proof (i.e, the proof of the tightness of $\{\mu_n\}_{n=1}^{\infty}$), the only condition required is that $g(t)$ (i.e., $\varphi_X(t)$ in the previous proof) is continuous at $t = 0$ and $g(0) = 1$. So the same proof can be used to show this corollary. □

Continuity theorem

- In the proof, the continuity of g was used to establish **tightness**. Hence, if $\{F_{X^n}\}_{n=1}^{\infty}$ is assumed tight in the first place, then the corollary apparently holds.

Corollary 2 Suppose a sequence of characteristic functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ has limits in every t, namely $\lim_{t\to\infty}\varphi_n(t)$ exists for every t.
Define

$$g(t) = \lim_{t\to\infty}\varphi_{X_n}(t).$$

Then if $\{\mu_n\}_{n=1}^{\infty}$ is tight, then there exists a probability measure μ such that

$$\mu_n \Rightarrow \mu, \text{ and } \mu \text{ has characteristic function } g$$

where μ_n is the probability measure corresponding to characteristic function $\varphi_n(\cdot)$.

Observation If $\{X_n\}_{n=1}^{\infty}$ is uniformly bounded, then $\{F_{X_n}\}_{n=1}^{\infty}$ is tight.

Continuity theorem

Example 26.2 Let X_n be uniformly distributed over $(-n, n)$.

Then $\varphi_{X_n}(t) = \int_{-n}^{n} e^{itx} \frac{1}{2n} dx = \frac{\sin(nt)}{nt}$.

Hence, the limit of $\varphi_{X_n}(t)$ exists, and is equal to:

$$g(t) = \lim_{n\to\infty} \varphi_{X_n}(t) = \lim_{n\to\infty} \frac{\sin(nt)}{nt} = \begin{cases} 0, & \text{if } t \neq 0; \\ \lim_{n\to\infty} \lim_{t\to 0} \cos(nt), & \text{if } t = 0 \end{cases} = \begin{cases} 0, & \text{if } t \neq 0; \\ 1, & \text{if } t = 0 \end{cases}$$

In this example, $\{F_{X_n}\}_{n=1}^{\infty}$ is not tight,
and as expected, $g(\cdot)$ is not continuous at $t = 0$.

Notably, the probability measures of X_n's *converges vaguely* to "all-zero measure."

Actually, $g(t)$ cannot be obtained by integrating e^{itx} with respect to the true all-zero-measure for $x \in \Re$, but a (bizarre) measure that is zero everywhere and integrates to 1 over the entire range. Hence, I put double quotation mark here to indicate it is not the usual all-zero measure.

Applying Fourier series to probabilities

Theorem Suppose the support of the distribution of random variable X is contained in $[0, 2\pi]$.

Then

$$\Pr[a < X \le b] = \lim_{m\to\infty} \int_a^b \sigma_m(t)dt,$$

if $\Pr[X = a] = \Pr[X = b] = 0$ and $0 < a < b < 2\pi$,
where

$$\sigma_m(t) = \frac{1}{2\pi m}\int_0^{2\pi} \frac{\sin^2[m(x-t)/2]}{\sin^2[(x-t)/2]} dF_X(x).$$

Proof: By Fubini's theorem,

$$\begin{aligned}
\int_a^b \sigma_m(t)dt &= \int_a^b \left(\frac{1}{2\pi m}\int_0^{2\pi} \frac{\sin^2[m(x-t)/2]}{\sin^2[(x-t)/2]} dF_X(x)\right) dt \\
&= \int_0^{2\pi} \frac{1}{2\pi m}\left(\int_a^b \frac{\sin^2[m(x-t)/2]}{\sin^2[(x-t)/2]} dt\right) dF_X(x) \\
&= \int_0^{2\pi} \frac{1}{2\pi m}\left(\int_a^b \frac{\sin^2[m(t-x)/2]}{\sin^2[(t-x)/2]} dt\right) dF_X(x) \\
&= \int_0^{2\pi} \frac{1}{2\pi m}\left(\int_{a-x}^{b-x} \frac{\sin^2(ms/2)}{\sin^2(s/2)} ds\right) dF_X(x).
\end{aligned}$$

Applying Fourier series to probabilities

By dominated convergence theorem, for any $\delta \in (0, 2\pi)$,

$$\lim_{m\to\infty} \frac{1}{2\pi m} \int_{\delta<|s|<2\pi} \frac{\sin^2(ms/2)}{\sin^2(s/2)} ds = \int_{\delta<|s|<2\pi} \frac{1}{2\pi \sin^2(s/2)} \left(\lim_{m\to\infty} \frac{\sin^2(ms/2)}{m} \right) ds = 0.$$

It can be shown (cf. the next slide) that for integer m,

$$\frac{1}{2\pi m} \int_{-\pi}^{\pi} \frac{\sin^2(ms/2)}{\sin^2(s/2)} ds = \frac{1}{2\pi m} \int_{-\pi}^{\pi} \left(\sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} e^{isk} \right) ds = \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} \int_{-\pi}^{\pi} e^{isk} ds$$

$$= \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} \frac{2\sin(k\pi)}{k} = \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} \frac{\sin(k\pi)}{k\pi} = 1 \quad \left(\text{because } \frac{\sin(k\pi)}{k\pi} = 0 \text{ except } k = 0\right)$$

Hence,

$$\frac{1}{2\pi m} \int_{|s|\le\delta} \frac{\sin^2(ms/2)}{\sin^2(s/2)} ds \stackrel{m\to\infty}{\longrightarrow} 1,$$

which implies that (note: $-2\pi < a - x < b - x < 2\pi$)

$$\lim_{m\to\infty} \frac{1}{2\pi m} \left(\int_{a-x}^{b-x} \frac{\sin^2(ms/2)}{\sin^2(s/2)} ds \right) = \begin{cases} 0, & \text{if } a - x > 0; \\ 1, & \text{if } a - x < 0 < b - x; \\ 0, & \text{if } b - x < 0 \end{cases} = \begin{cases} 0, & \text{if } x < a \text{ or } x > b; \\ 1, & \text{if } a < x < b. \end{cases}$$

Applying Fourier series to probabilities

$$
\begin{aligned}
\sum_{\ell=0}^{m-1}\sum_{k=-\ell}^{\ell} e^{isk} &= \sum_{\ell=0}^{m-1}\frac{e^{-is\ell}(1-e^{is(2\ell+1)})}{1-e^{is}} \\
&= \frac{1}{1-e^{is}}\left(\sum_{\ell=0}^{m-1} e^{-is\ell} - \sum_{\ell=0}^{m-1} e^{is(\ell+1)}\right) \\
&= \frac{1}{1-e^{is}}\left(\frac{1-e^{-ism}}{1-e^{-is}} - \frac{e^{is}(1-e^{ism})}{1-e^{is}}\right) \\
&= \frac{1}{1-e^{is}}\left(\frac{e^{-is(m-1)}(1-e^{ism})}{1-e^{is}} - \frac{e^{is}(1-e^{ism})}{1-e^{is}}\right) \\
&= \frac{e^{-ism}(1-e^{ism})^2}{e^{is}(1-e^{is})^2} = \frac{\sin^2(ms/2)}{\sin^2(s/2)}.
\end{aligned}
$$

As $\Pr[X=a]=\Pr[X=b]=0$, we obtain from dominated convergence theorem that

$$
\lim_{m\to\infty}\int_a^b \sigma_m(t)dt = \int_0^{2\pi}\left(\lim_{m\to\infty}\frac{1}{2\pi m}\int_{a-x}^{b-x}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds\right)dF_X(x) = \Pr[a<X<b].
$$

□

Applying Fourier series to probabilities

Discussions:

- By following from the previous theorem, the Fourier coefficients are defined by:

$$c_m = \int_0^{2\pi} e^{imx} dF_X(x) \text{ for } m = 0, \pm 1, \pm 2, \cdots .$$

- $\{c_m\}_{m=0,\pm1,\pm2,\ldots}$ can be viewed as the *values of the characteristic function for integer arguments.*
  In other words,

$$c_m = \varphi_X(m),$$

  where

$$\varphi_X(t) = \int_0^{2\pi} e^{itx} dF_X(x).$$

- Notably,

$$\begin{aligned}\sigma_m(t) &= \frac{1}{2\pi m}\int_0^{2\pi} \frac{\sin^2[m(x-t)/2]}{\sin^2[(x-t)/2]} dF_X(x) \\ &= \frac{1}{2\pi m}\int_0^{2\pi} \sum_{\ell=0}^{m-1}\sum_{k=-\ell}^{\ell} e^{i(x-t)k} dF_X(x) = \frac{1}{2\pi m}\sum_{\ell=0}^{m-1}\sum_{k=-\ell}^{\ell} c_k e^{-itk}.\end{aligned}$$

Applying Fourier series to probabilities

- Hence, the distribution of a random variable with support contained in $[0, 2\pi]$ can be uniquely determined by the **samples** of $\varphi_X(t)$, namely $\{\varphi_X(m)\}_{m=0,\pm 1,\pm 2,\ldots}$.
  - **Trivial Extension:** The distribution of a random variable with bounded support can be uniquely determined by the samples of its characteristic function (if with a properly selected sampling period).
  - A drawback of using samples to determine the distribution with bounded support is that the probability masses at two end points are undetermined.

    In other words, using the above theorem as an example, we can have two distinct distributions with equal positive $\Pr[X = 0] + \Pr[X = 2\pi] > 0$, but their "samples" are identical.

    Example $\Pr[X = 0] = \Pr[Y = 2\pi] = 1$.
    Then
    $$\varphi_X(t) = E[e^{itX}] = 1 \text{ and } \varphi_Y(t) = E[e^{itY}] = e^{i2\pi t}.$$
    However, for every integer m,
    $$1 = \varphi_X(m) = \varphi_Y(m) = e^{i2\pi m} = 1.$$

    Observation Distributions with the same $\Pr[X = 0] + \Pr[X = 2\pi]$ are indistinguishable using the approach of Fourier series.

Applying Fourier series to probabilities

Corollary Suppose the support of the distribution of random variable X is contained in $[0, 2\pi]$. Then for $0 < a < b < 2\pi$,

$$\frac{1}{2}\Pr[X = a] + \Pr[a < X < b] + \frac{1}{2}\Pr[X = b] = \lim_{m\to\infty}\int_a^b \sigma_m(t)dt.$$

Proof: Since $\sin^2(ms/2)/\sin^2(s/2)$ is a bounded even function,

$$\frac{1}{2\pi m}\int_{-\pi}^{0}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds = \frac{1}{2\pi m}\int_{0}^{\pi}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds.$$

Hence,

$$\frac{1}{2\pi m}\int_{-\pi}^{0}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds = \frac{1}{2\pi m}\int_{0}^{\pi}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds = \frac{1}{2},$$

which implies that

$$\lim_{m\to\infty} \frac{1}{2\pi m}\left(\int_{a-x}^{b-x} \frac{\sin^2(ms/2)}{\sin^2(s/2)}ds\right) = \begin{cases} 0, & \text{if } a-x>0; \\ \frac{1}{2}, & \text{if } a-x=0; \\ 1, & \text{if } a-x<0<b-x; \\ \frac{1}{2}, & \text{if } b-x=0; \\ 0, & \text{if } b-x<0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } x<a \text{ or } x>b; \\ \frac{1}{2}, & \text{if } x=a \text{ or } x=b; \\ 1, & \text{if } a<x<b. \end{cases}$$

Accordingly, the corollary holds. □

Theorem Suppose $\{X_n\}_{n=1}^{\infty}$ have supports in $[0, 2\pi]$, and suppose the mth Fourier coefficient $c_m(n)$ of X_n converges to c_m. Then

$$X_n \Rightarrow X,$$

where X is the distribution determined through $\{c_m\}_{m \text{ integer}}$.
The distribution of X is unique except possibley in the way the mass on $\{0, 2\pi\}$ is split on the points of 0 and 2π.

Proof: Note that a sequence of random variables is tight, if their supports are uniformly bounded. □

Uniformly distributed modulo 1

Give a sequence of real numbers $x_1, x_2, x_3, \ldots$.

Let the probability measure μ_n put masses $1/n$ at points $2\pi\,(x_k - \lfloor x_k \rfloor)$.

Let the probability measure μ be uniformly distributed over $(0, 2\pi]$.

Hence,

$$c_m(n) = \int_0^{2\pi} e^{imx}\mu_n(dx) = \frac{1}{n}\sum_{k=1}^{n} e^{i2\pi m(x_k - \lfloor x_k \rfloor)},$$

and

$$c_m = \int_0^{2\pi} e^{imx}\mu(dx) = \frac{1}{2\pi}\int_0^{2\pi} e^{imx}dx = \begin{cases} 1, & \text{if } m = 0; \\ 0, & \text{if } m \neq 0. \end{cases}$$

Therefore, $\mu_n \Rightarrow \mu$, if

$$\text{for } m \neq 0, \quad \frac{1}{n}\sum_{k=1}^{n} e^{i2\pi m(x_k - \lfloor x_k \rfloor)} \stackrel{n\to\infty}{\longrightarrow} 0.$$

This is named **Weyl's criterion**.

Uniformly distributed modulo 1

Now if $x_k = k\theta$, where θ is irrational, then $e^{i2\pi m\theta} \neq 1$ for $m \neq 0$, and

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} e^{i2\pi m(x_k - \lfloor x_k \rfloor)} &= \frac{1}{n}\sum_{k=1}^{n} e^{i2\pi m(k\theta - \lfloor k\theta \rfloor)} \\
&= \frac{1}{n}\sum_{k=1}^{n} e^{i2\pi m k\theta} \\
&= \frac{1}{n} e^{i2\pi m\theta} \frac{1 - e^{i2\pi m\theta n}}{1 - e^{i2\pi m\theta}} \\
&= \frac{1}{n} e^{i2\pi m\theta} \frac{e^{i\pi m\theta n}\left(e^{-i\pi m\theta n} - e^{i\pi m\theta n}\right)}{e^{i\pi m\theta}\left(e^{-i\pi m\theta} - e^{i\pi m\theta}\right)} \\
&= e^{i\pi m\theta(n+1)} \frac{\sin(\pi m\theta n)}{n\sin(\pi m\theta)} \\
&\stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

$\square$

- **Question:** Is (inverse) Fourier transform always continuous, just like the characteristic function?

  Answer: No. Think of the (inverse) Fourier transform of $\sin(x)/x$.

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{itx} dx = \begin{cases} 0, & \text{if } |t| > 1; \\ \frac{\pi}{2}, & \text{if } |t| = 1; \\ \pi, & \text{if } |t| < 1. \end{cases}$$

  If $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then the (inverse) Fourier transform is uniformly continuous.

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{i(t+h)x} f(x) dx - \int_{-\infty}^{\infty} e^{itx} f(x) dx \right| &= \left| \int_{-\infty}^{\infty} (e^{ihx} - 1) e^{itx} f(x) dx \right| \\ &\leq \int_{-\infty}^{\infty} \left| (e^{ihx} - 1) \right| \cdot \left| e^{itx} \right| \cdot |f(x)| dx \\ &= \int_{-\infty}^{\infty} \left| (e^{ihx} - 1) \right| |f(x)| dx \\ &= \left(\int_{-\infty}^{\infty} |f(x)| dx \right) \int_{-\infty}^{\infty} \left| (e^{ihx} - 1) \right| dF(x), \end{aligned}$$

  where $dF(x) = |f(x)| dx / \left(\int_{-\infty}^{\infty} |f(x)| dx \right)$.

Fourier Transform

- **Question:** Does (inverse) Fourier transform has Taylor expansion?

  Answer: This section learns us that if

  1. $\lim_{n\to\infty} \frac{|t|^n \int_{\Re} |x|^n |f(x)| dx}{n!} = 0$, or
  2. $\int_{\Re} e^{|t||x|} |f(x)| dx < \infty$,

  then Fourier transform has Taylor expansion at $t = 0$.

  We also learn that even if Fourier transform does not equal its infinite Taylor sum, we can still express it in partial Taylor sum up to order n, if

$$\int_{\Re} |x|^n |f(x)| dx < \infty.$$

Fourier Transform

- **Question:** Does (inverse) Fourier transform has derivatives?

  Answer: If
  $$\int_{\Re} |x|^n |f(x)| dx < \infty,$$
  then Fourier transform has the kth derivatives at $t = 0$ for all $k \leq n$.

More can be established through the lessen from this section!