Section 21

Expected Values

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Expected value as integral

Definition (expected value) The expected value of a random variable X is the integral of X with respect to its cdf. I.e.,

$$E[X] = E[X^+] - E[X^-] = \int_0^\infty x dF_X(x) - \int_{-\infty}^0 (-x) dF_X(x) = \int_{-\infty}^\infty x dF_X(x).$$

Properties of expected value

1. $E[X]$ exists if, and only if, at least one of $E[X^+]$ and $E[X^-]$ is finite.

2. X is integrable, if $E[|X|] < \infty$.

3. For $\mathcal{B}/\mathcal{B}$-measurable (or simply $\mathcal{B}$-measurable) function g,

$$\begin{aligned} E[g(X)] &= E[g(X)^+] - E[g(X)^-] \\ &= \int_{\{x \in \Re : g(x) \geq 0\}} g(x) dF_X(x) - \int_{\{x \in \Re : g(x) < 0\}} [-g(x)] dF_X(x) \\ &= \int_{-\infty}^\infty g(x) dF_X(x). \end{aligned}$$

Absolute moments

Definition (absolute moments) The kth absolute moment of X is:

$$E\left[|X|^k\right] = \int_{-\infty}^{\infty} |x|^k dF_X(x).$$

Properties of absolute moments

1. The absolute moment always exists.
2. If the kth absolute moment is finite, then the jth absolute moment is finite for any $j \leq k$.

   Proof: It can be easily proved by $|x|^j \leq 1 + |x|^k$ for $j \leq k$. □

Moments

Definition (moments) The kth moment of X is:

$$\begin{aligned} E\left[X^k\right] &= \int_{-\infty}^{\infty} x^k dF_X(x) \\ &= \int_{-\infty}^{\infty} \max\{x^k, 0\} dF_X(x) - \int_{-\infty}^{\infty} \left(-\min\{x^k, 0\}\right) dF_X(x) \\ &= E[(X^k)^+] - E[(X^k)^-]. \end{aligned}$$

Properties of moments

1. $E[X^k]$ exists if, and only if, at least one of $E[(X^k)^+]$ and $E[(X^k)^-]$ is finite.

2. X^k is integrable, if $E[|X^k|] < \infty$.

3. For $\mathcal{B}/\mathcal{B}$-measurable (or simply $\mathcal{B}$-measurable) function g,

$$\begin{aligned} E[g(X^k)] &= E[g(X^k)^+] - E[g(X^k)^-] \\ &= \int_{\{x \in \Re : g(x^k) \geq 0\}} g(x^k) dF_X(x) - \int_{\{x \in \Re : g(x^k) < 0\}} [-g(x^k)] dF_X(x) \\ &= \int_{-\infty}^{\infty} g(x^k) dF_X(x). \end{aligned}$$

Example 21.1 (moments of standard normal) The pdf of a standard normal distribution is equal to:

$$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Integration by parts shows that:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}x^k e^{-x^2/2}dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}(k-1)x^{k-2}e^{-x^2/2}dx,$$

or equivalently,

$$E[X^k] = (k-1)E[X^{k-2}].$$

As a result,

$$E[X^k] = \begin{cases} 0, & \text{for } k \text{ odd;} \\ 1 \times 3 \times 5 \times \cdots \times (k-1), & \text{for } k \text{ even.} \end{cases}$$

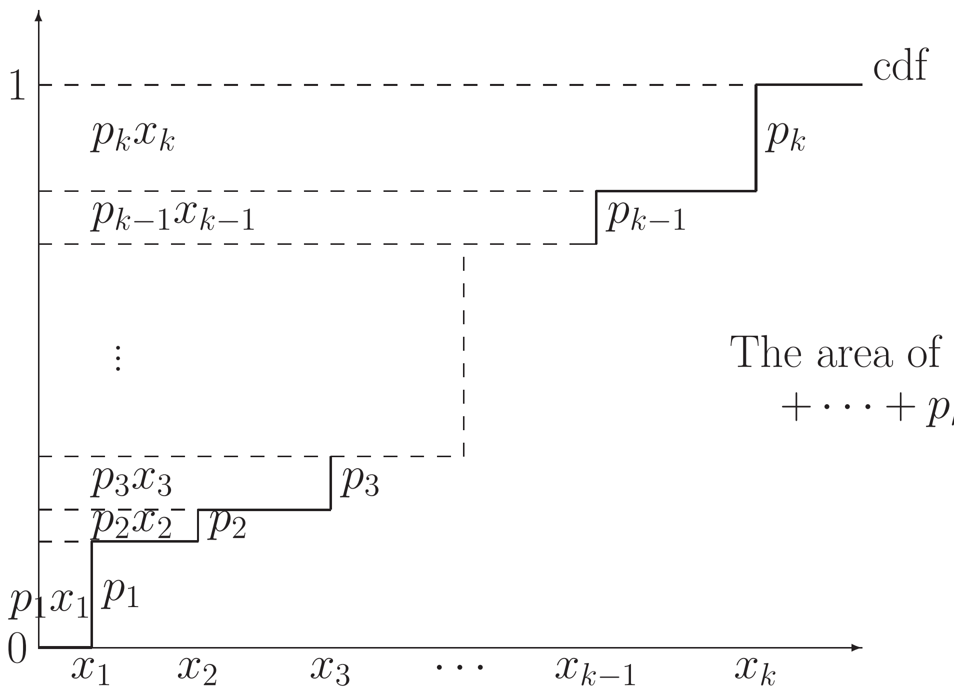
Computation of mean

Theorem For non-negative random variable X,

$$E[X] = \int_0^\infty \Pr[X > t]dt = \int_0^\infty \Pr[X \geq t]dt.$$

In other words, the area of ccdf (complementary cdf) is the mean. (The theorem is valid even if $E[X] = \infty$.)(This is an extension of the law of large numbers when empirical distribution is concerned)!

Geometric interpretation Suppose $\Pr[X = x_i] = p_i$ for $1 \leq i \leq k$.



The area of ccdf $= p_1 x_1 + p_2 x_2 + p_3 x_3 + \cdots + p_{k-1} x_{k-1} + p_k x_k = E[X]$

Note that it is always true that $\int_0^\infty x dF_X(x) = \int_{0^+}^\infty x dF_X(x)$ but it is possible $\int_\alpha^\infty x dF_x(x) \neq \int_{\alpha^+}^\infty x dF_X(x)$!

Computation of mean

Theorem For $\alpha \geq 0$,

$$\int_{\alpha^+}^\infty x dF_X(x) = \alpha \Pr[X > \alpha] + \int_\alpha^\infty \Pr[X > t] dt.$$

Proof: Let $Y = X \times I_{[X>\alpha]}$, where $I_{[X>\alpha]} = 1$ if $X > \alpha$, and zero, otherwise. Hence, $Y = 0$ for $X \leq \alpha$, and $Y = X$ for $X > \alpha$. Consequently,

$$\begin{aligned}
\int_{\alpha^+}^\infty x dF_X(x) = E[Y] &= \int_0^\infty \Pr[Y > t] dt \\
&= \int_0^\alpha \Pr[Y > t] dt + \int_\alpha^\infty \Pr[Y > t] dt \\
&= \int_0^\alpha \Pr[X > \alpha] dt + \int_\alpha^\infty \Pr[X > t] dt \\
&= \alpha \Pr[X > \alpha] + \int_\alpha^\infty \Pr[X > t] dt.
\end{aligned}$$

□

- The empirical approximation of $\Pr[X > t]$ (or $\Pr[X \leq t]$) is more easily obtained than $dF(x)$. With the above result, $E[X]$ (or $E[Y]$) can be established directly from $\Pr[X > t]$.

Inequalities regarding moments

Lemma (Markov's inequality) For any $k > 0$ (and implicitly $\alpha > 0$),

$$\Pr[|X| \geq \alpha] \leq \frac{1}{\alpha^k} E[|X|^k].$$

Proof: The below inequality is valid for any $x \in \Re$:

$$|x|^k \geq \alpha^k \cdot \mathbf{1}\{|x|^k \geq \alpha^k\} \tag{21.1}$$

Hence,

$$\underline{E[|X|^k]} = \int_{-\infty}^{\infty} |x|^k dF_X(x) \geq \alpha^k \cdot \int_{-\infty}^{\infty} \mathbf{1}\{|x|^k \geq \alpha^k\} dF_X(x) = \alpha^k \underline{\Pr[|X| \geq \alpha]}.$$

Equality holds if, and only if, equality in (21.1) is true with probability 1. I.e.,

$$\Pr\left[|X|^k = \alpha^k \cdot \mathbf{1}\{|X|^k \geq \alpha^k\}\right] = 1,$$

or equivalently, $\Pr[|X| = 0 \text{ or } \alpha] = 1$. □

Chebyshev-Bienaymé inequality

Lemma (Chebyshev-Bienaymé inequality) For $\alpha > 0$,

$$\Pr[|X - E[X]| \geq \alpha] \leq \frac{1}{\alpha^2}\text{Var}[X].$$

Proof: By Markov's inequality with $k = 2$, we have:

$$\Pr[|X - E[X]| \geq \alpha] \;\leq\; \frac{1}{\alpha^2}E[|X - E[X]|^2].$$

Equality holds if, and only if,

$$\Pr[|X - E[X]| = 0] + \Pr\left[|X - E[X]| = \alpha\right] = 1,$$

which implies that

$$\Pr\left[X = E[X] + \alpha\right] = \Pr\left[X = E[X] - \alpha\right] = p$$

and

$$\Pr[X = E[X]] = 1 - 2p$$

for $\alpha > 0$. $\square$

Jensen's inequality

Definition (convexity) A function $\varphi(x)$ is said to be *convex* over an interval (a, b) if for every $x_1, x_2 \in (a, b)$ and $0 \leq \lambda \leq 1$,

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda\varphi(x_1) + (1 - \lambda)\varphi(x_2).$$

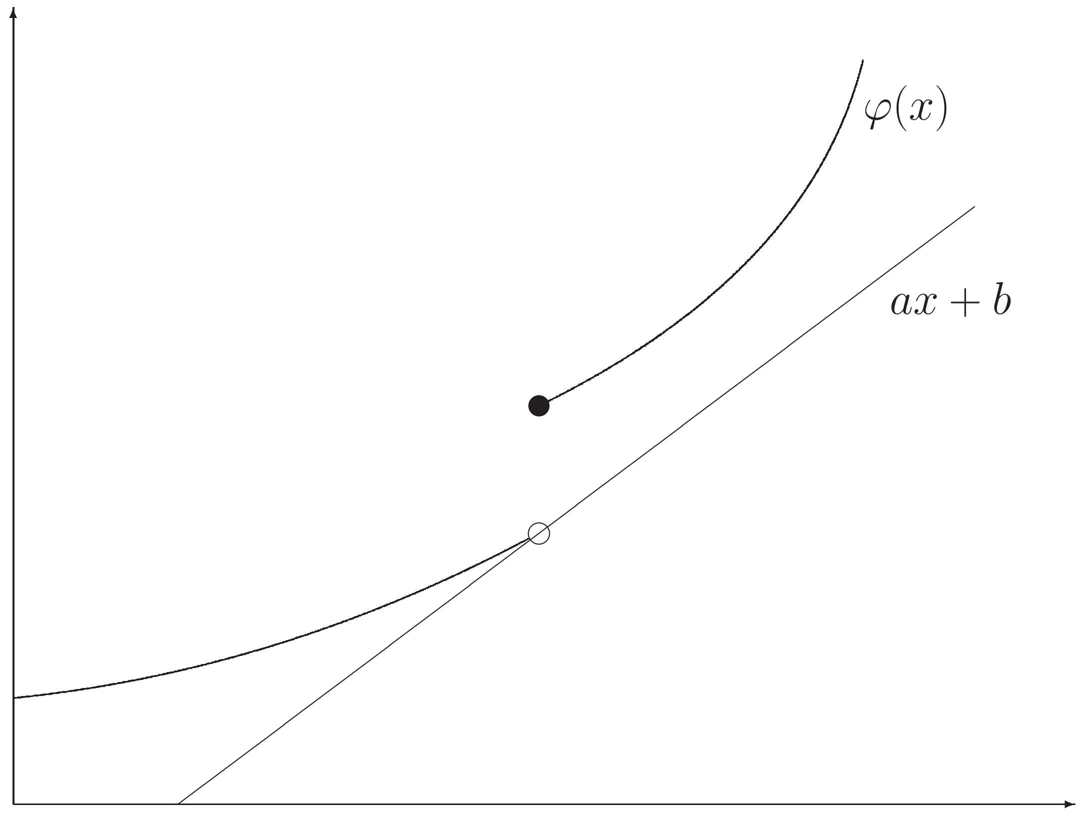
Furthermore, a function φ is said to be *strictly convex* if equality holds only when $\lambda = 0$ or $\lambda = 1$. (Can we replace (a, b) by a real set $\mathcal{X}$?)

Definition (support line) A line $y = ax + b$ is said to be a support line of function $\varphi(x)$ if among all lines of the same slope a, it is the largest one satisfying $ax + b \leq \varphi(x)$ for every x.

- A support line $ax + b$ may not necessarily intersect with $\varphi(\cdot)$. In other words, it is possible that no x_0 satisfies $ax_0 + b = \varphi(x_0)$.
- However, the existence of intersection between function $\varphi(\cdot)$ and its support line is guaranteed, if $\varphi(\cdot)$ is convex.

Jensen's inequality

An example that no intersection exists for a function and its support line



Jensen's inequality

Lemma (Jensen's inequality) Suppose that function $\varphi(\cdot)$ is convex on the domain $\mathcal{X}$ of X. (Implicitly, $E[X] \in \mathcal{X}$.) Then

$$\varphi(E[X]) \leq E[\varphi(X)].$$

Proof: Let $ax + b$ be a support line through the point $(E[X], \varphi(E[X]))$. Thus, over the domain $\mathcal{X}$ of $\varphi(x)$,

$$ax + b \leq \varphi(x).$$

If equality holds in $\mathcal{X}$ in this step, then equality remains true for the subsequent steps.

By taking the expectation value of both sides, we obtain

$$a \cdot E[X] + b \leq E[\varphi(X)],$$

but we know that $a \cdot E[X] + b = \varphi(E[X])$. Consequently,

$$\varphi(E[X]) \leq E[\varphi(X)].$$

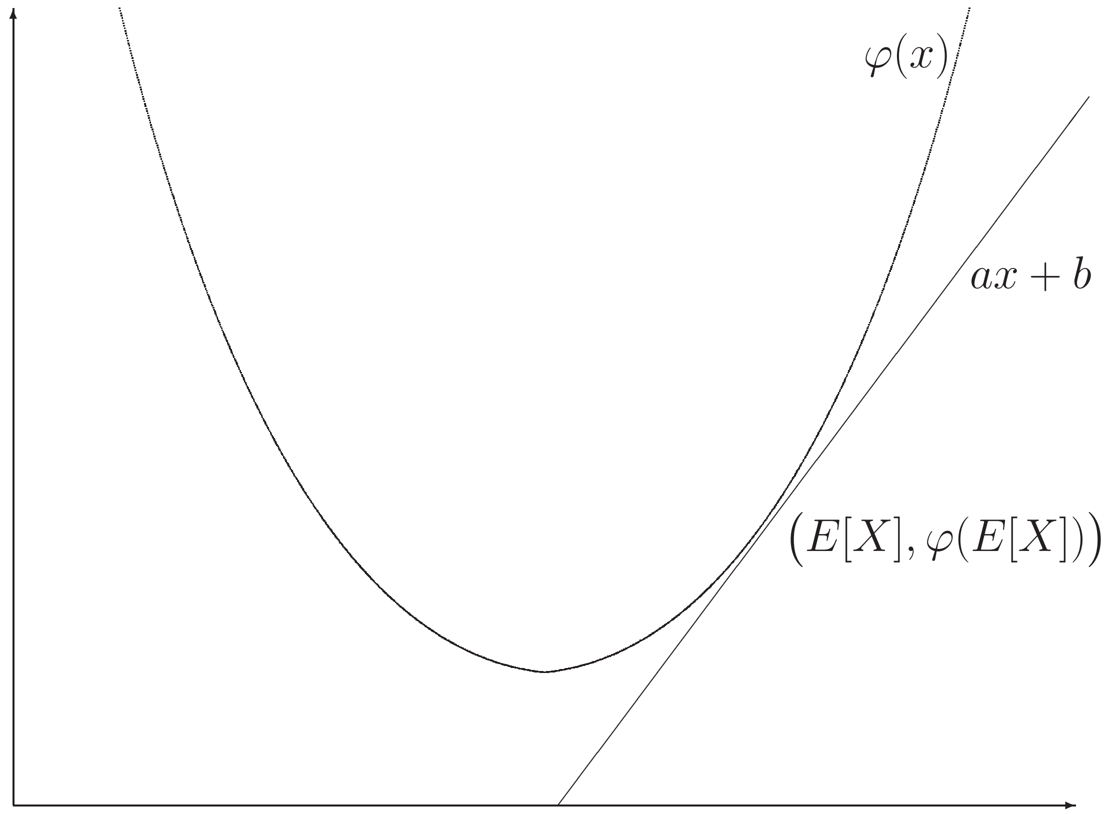
Equality holds if, and only if, there exist a and b such that $aE[X] + b = \varphi(E[X])$ and

$$\Pr\big(\{x \in \mathcal{X} : ax + b = \varphi(x)\}\big) = 1.$$

$\square$

Jensen's inequality

The support line $y = ax + b$ of the convex function $\varphi(x)$.



Hölder's inequality

Lemma (Hölder's inequality) For $p > 1, q > 1$ and $1/p + 1/q = 1$,

$$E[|XY|] \leq E^{1/p}[|X|^p]E^{1/q}[|Y|^q].$$

Proof: Since the inequality is trivially valid, if $E^{1/p}[|X|^p]E^{1/q}[|Y|^q] = 0$. Without loss of generality, assume $E^{1/p}[|X|^p]E^{1/q}[|Y|^q] > 0$.

- $\exp\{x\}$ is a convex function in x. Hence, by Jensen's inequality,

$$\exp\left\{\frac{1}{p}s + \frac{1}{q}t\right\} \leq \frac{1}{p}\exp\{s\} + \frac{1}{q}\exp\{t\}.$$

Since e^x is strictly convex, equality holds iff $s = t$.

- Let $a = \exp\{s/p\}$ and $b = \exp\{t/q\}$. Then the above inequality becomes:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q,$$

Equality holds iff $a^p = b^q$.

whose validity is not restricted to positive a and b but to non-negative a and b.

- By letting $a = |x|/E^{1/p}[|X|^p]$ and $b = |y|/E^{1/q}[|Y|^q]$, we obtain:

$$\frac{|xy|}{E^{1/p}[|X|^p]E^{1/q}[|Y|^q]} \leq \frac{1}{p}\frac{|x|^p}{E[|X|^p]} + \frac{1}{q}\frac{|y|^q}{E[|Y|^q]}.$$

Equality holds if, and only if, $\Pr\left[\frac{|X|^p}{E[|X|^p]} = \frac{|Y|^q}{E[|Y|^q]}\right] = 1.$

Hölder's inequality

Taking the expectation values of both sides yields:

$$\frac{E[|XY|]}{E^{1/p}[|X|^p]E^{1/q}[|Y|^q]} \le \frac{1}{p}\frac{E[|X|^p]}{E[|X|^p]} + \frac{1}{q}\frac{E[|Y|^q]}{E[|Y|^q]} = \frac{1}{p} + \frac{1}{q} = 1. \qquad \square$$

Lemma (Hölder's inequality) For $p > 1, q > 1$ and $1/p + 1/q = 1$,

$$E[|XY|] \le E^{1/p}[|X|^p]E^{1/q}[|Y|^q].$$

Equality holds if, and only if,

$$\Pr\left[\frac{|X|^p}{E[|X|^p]} = \frac{|Y|^q}{E[|Y|^q]}\right] = 1 \text{ or } \Pr[X = 0] = 1 \text{ or } \Pr[Y = 0] = 1.$$

Example. $p = q = 2$ and

	$Y = 0$	$Y = 1$
$X = 0$	p_{00}	p_{01}
$X = 1$	p_{10}	p_{11}

$$\begin{aligned} E[|XY|] &= p_{11} \\ &= p_{11}^{1/2} p_{11}^{1/2} \\ &\le (p_{10} + p_{11})^{1/2}(p_{01} + p_{11})^{1/2} \\ &= E^{1/2}[|X|^2]E^{1/2}[|Y|^2] \end{aligned}$$

with equality holding iff $p_{10} = p_{01} = 0$ or $p_{10} = p_{11} = 0$ or $p_{01} = p_{11} = 0$. $\square$

Cauchy-Schwartz's inequality

Suppose $E[|X|] > 0$ and $E[|Y| > 0$.

Lemma (Hölder's inequality) For $p > 1, q > 1$ and $1/p + 1/q = 1$,

$$E[|XY|] \leq E^{1/p}[|X|^p]E^{1/q}[|Y|^q].$$

Equality holds if, and only if, there exists a such that $\Pr[|X|^p = a|Y|^q] = 1$.

Lemma (Cauchy-Schwartz's inequality)

$$E[|XY|] \leq E^{1/2}[X^2]E^{1/2}[Y^2].$$

Equality holds if, and only if, there exists a such that $\Pr[X^2 = aY^2] = 1$.

Proof: A special case of Hölder's inequality with $p = q = 2$.
Equality holds if, and only if,

$$\Pr\left[X^2 = aY^2\right] = 1$$

for some a. □

Lyapounov's inequality

Lemma (Lyapounov's inequality) For $0 < \alpha < \beta$,

$$E^{1/\alpha}[|Z|^\alpha] \le E^{1/\beta}[|Z|^\beta].$$

Equality holds if, and only if, $\Pr[|Z| = a] = 1$ for some a.

Proof: Letting $X = |Z|^\alpha$, $Y = 1$, $p = \beta/\alpha$ and $q = \beta/(\beta - \alpha)$ in Hölder's inequality yields:

$$E[|Z|^\alpha] \le E^{\alpha/\beta}\left[(|Z|^\alpha)^{\beta/\alpha}\right] E^{(\beta-\alpha)/\beta}\left[1^{\beta/(\beta-\alpha)}\right] = E^{\alpha/\beta}[|Z|^\beta].$$

Equality holds if, and only if,

$$\Pr\left[(|Z|^\alpha)^{\beta/\alpha} = a\right] = \Pr\left[|Z|^\beta = a\right] = \Pr\left[|Z| = a^{1/\beta}\right] = 1$$

for some a (including $a = 0$). □

- Notably, in the statement of the lemma, β is strictly larger than α.
- It is certain that if $\alpha = \beta$, the inequality automatically becomes an equality.

Joint integrals

Lemma For $\mathcal{B}^k/\mathcal{B}$-measurable (or simply $\mathcal{B}^k$-measurable) function g and k-dimensional random vector $\boldsymbol{X}$,

$$E[g(\boldsymbol{X})] = \int_{\Re^k} g(x^k) dF_{\boldsymbol{X}}(x^k),$$

if one of $E[(g(\boldsymbol{X}))^+]$ and $E[(g(\boldsymbol{X}))^-]$ is finite.

Definition (covariance) The *covariance* of two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ is:

$$\begin{aligned}
\text{Cov}[\boldsymbol{X}, \boldsymbol{Y}] &= E\left[(\boldsymbol{X} - E[\boldsymbol{X}])(\boldsymbol{Y} - E[\boldsymbol{Y}])^T\right] \\
&= E\left[\begin{bmatrix} X_1 - E[X_1] \\ X_2 - E[X_2] \\ \vdots \\ X_k - E[X_k] \end{bmatrix} \begin{bmatrix} Y_1 - E[Y_1] & Y_2 - E[Y_2] & \cdots & Y_\ell - E[Y_\ell] \end{bmatrix}\right] \\
&= \begin{bmatrix} (X_1 - E[X_1])(Y_1 - E[Y_1]) & \cdots & (X_1 - E[X_1])(Y_\ell - E[Y_\ell]) \\ \vdots & \cdots & \vdots \\ (X_k - E[X_k])(Y_1 - E[Y_1]) & \cdots & (X_k - E[X_k])(Y_\ell - E[Y_\ell]) \end{bmatrix}_{k \times \ell}
\end{aligned}$$

where "T" represents vector transpose operation, if one of $E[((X_i - E[X_i])(Y_j - E[Y_j]))^+]$ and $E[((X_i - E[X_i])(Y_j - E[Y_j]))^-]$ is finite for every i, j.

Joint integrals

Definition (uncorrelated) $\boldsymbol{X}$ and $\boldsymbol{Y}$ is uncorrelated, if

$$\mathrm{Cov}[\boldsymbol{X},\boldsymbol{Y}] = \mathbf{0}_{k\times\ell}.$$

Definition (independence) $\boldsymbol{X}$ and $\boldsymbol{Y}$ is independent, if

$$\begin{aligned} & \Pr\left[(X_1 \le x_1 \wedge \cdots \wedge X_k \le x_k) \wedge (Y_1 \le y_1 \wedge \cdots \wedge Y_\ell \le y_\ell)\right] \\ = \ & \Pr\left[X_1 \le x_1 \wedge \cdots \wedge X_k \le x_k\right] \Pr\left[Y_1 \le y_1 \wedge \cdots \wedge Y_\ell \le y_\ell\right]. \end{aligned}$$

Lemma (integrability of product) For independent $X_1, X_2, \ldots, X_k$, if each of X_i is integrable, so is $X_1X_2\cdots X_k$, and

$$E[X_1X_2\cdots X_k] = E[X_1]E[X_2]\cdots E[X_k].$$

Lemma (sum of variance for pair-wise independent samples) If $X_1, X_2, \ldots, X_k$ are pair-wise independent and integrable,

$$\mathrm{Var}[X_1 + X_2 + \cdots + X_k] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_k].$$

- Notably, *pair-wise independence* does not imply *complete independence*.

Example (only pair-wise independence) Toss a fair coin twice, and assume independence. Define

$$X = \begin{cases} 1, & \text{if head appears on the first toss;} \\ 0, & \text{otherwise,} \end{cases}$$

$$Y = \begin{cases} 1, & \text{if head appears on the second toss;} \\ 0, & \text{otherwise,} \end{cases}$$

$$Z = \begin{cases} 1, & \text{if exactly one head and one tail appear on the two tosses;} \\ 0, & \text{otherwise,} \end{cases}$$

Then $\Pr[X = 1 \wedge Y = 1 \wedge Z = 1] = 0$;
but $\Pr[X = 1] \Pr[Y = 1] \Pr[Z = 1] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.
So X, Y and Z are not independent.

Obviously, $X \perp\!\!\!\perp Y$. In addition,

$$\left.\begin{aligned} \Pr[X = 1|Z = 1] &= \frac{\Pr[X = 1 \wedge Z = 1]}{\Pr[Z = 1]} = \frac{1/4}{1/2} = \frac{1}{2} = \Pr[X = 1] \\ \Pr[X = 1|Z = 0] &= \frac{\Pr[X = 1 \wedge Z = 0]}{\Pr[Z = 0]} = \frac{1/4}{1/2} = \frac{1}{2} = \Pr[X = 1] \end{aligned}\right\} \Rightarrow X \perp\!\!\!\perp Z.$$

One can similarly show (or by symmetry) that $Y \perp\!\!\!\perp Z$.

Joint integrals

Example (con't) By

$$\left.\begin{array}{l} \text{head head} \Rightarrow X+Y+Z=2 \\ \text{head tail} \Rightarrow X+Y+Z=2 \\ \text{tail head} \Rightarrow X+Y+Z=2 \\ \text{tail tail} \Rightarrow X+Y+Z=0 \end{array}\right\} \Rightarrow \begin{array}{l} \mathrm{Var}[X+Y+Z] \\ = \frac{3}{4}(2-3/2)^2 + \frac{1}{4}(0-3/2)^2 = \frac{3}{4} \end{array}$$

This is equal to:

$$\mathrm{Var}[X] + \mathrm{Var}[Y] + \mathrm{Var}[Z] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

This result matches that "The variance of sum equals the sum of variances" holds for pair-wise independent random variables.

Moment generating function

Definition (moment generating function) The moment generating function of X is defined as:

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} dF_X(x),$$

for all t for which this is finite.

- If $M_X(t)$ is defined (i.e., finite) throughout an interval $(-t_0, t_0)$, where $t_0 > 0$. then

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k].$$

  In other words, $M_X(t)$ has a Taylor expansion about $t = 0$ with positive radius of convergence if it is defined in some $(-t_0, t_0)$.

- In case that $M_X(t)$ has a Taylor expansion about $t = 0$ with positive radius of convergence, the moment of X can be computed by the derivatives of $M_X(t)$ through:

$$M^{(k)}(0) = E[X^k].$$

- If $M_{X_i}(t)$ is defined throughout an interval $(-t_0, t_0)$ for each i, and $X_1, X_2, \ldots, X_n$ are independent, then the moment generating function of $X_1 + \cdots + X_n$ is also defined on $(-t_0, t_0)$, and is equal to $\prod_{i=1}^{n} M_{X_i}(t)$.

Moment generating function

Example (random variable whose moment generating function is defined only at zero) The pdf of a Cauchy distribution is

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

For $t \neq 0$, the integral

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{tx}\frac{1}{\pi(1+x^2)}dx &= \int_0^{\infty} \left(e^{tx}+e^{-tx}\right)\frac{1}{\pi(1+x^2)}dx \\
&\geq \int_0^{\infty} e^{|t|x}\frac{1}{\pi(1+x^2)}dx \\
&\geq \int_0^{\infty} \frac{|t|x}{\pi(1+x^2)}dx \quad \text{(by } e^x \geq 1+x \geq x \text{ for } x \geq 0\text{)} \\
&\geq \int_1^{\infty} \frac{|t|x}{\pi(1+x^2)}dx \geq \int_1^{\infty} \frac{|t|x}{\pi(x^2+x^2)}dx = \frac{|t|}{2\pi}\int_1^{\infty}\frac{1}{x}dx = \infty.
\end{aligned}$$

So the moment generating function is not defined for any $t \neq 0$.

The Cauchy distribution is indeed the Student's T-distribution with 1 degree of freedom.

Example (Student's T-distribution with n degree of freedom)
This distribution has moments of order $\leq n-1$ but any higher moments either do not exist or are infinity.

Some books considers infinite moments as "non-existence". So they write "This distribution has moments of order $\leq n-1$ but no higher moments exist."

Let $X, Y_1, Y_2, \ldots, Y_n$ be i.i.d. with standard normal marginal distribution. Then

$$T_n = \frac{X\sqrt{n}}{\sqrt{Y_1^2 + \cdots + Y_n^2}}$$

is called the Student's t-distribution (on $\Re$) with n degree of freedom.

- The numerator $X\sqrt{n}$ has a normal density with mean 0 and variance n.
- $\boldsymbol{\chi}^2 = Y_1^2 + \cdots + Y_n^2$ is a *chi-square distribution* with n degree of freedom, and has density $f_{1/2,n/2}(y)$, where

$$f_{\alpha,\nu}(x) = \frac{1}{\Gamma(\nu)}\alpha^\nu x^{\nu-1}e^{-\alpha x} \quad \text{on } [0,\infty)$$

  is the *gamma density* (or sometimes named *Erlangian density* when ν is a positive integer) with parameters $\nu > 0$ and $\alpha > 0$, and $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the gamma function.

(closure under convolutions for gamma density) $f_{\alpha,\mu} * f_{\alpha,\nu} = f_{\alpha,\mu+\nu}$.

Proof:

$$
\begin{aligned}
(f_{\alpha,\mu} * f_{\alpha,\nu})(x) &= \int_0^\infty f_{\alpha,\mu}(x-y) f_{\alpha,\nu}(y)dy \\
&= \frac{\alpha^{\mu+\nu}}{\Gamma(\mu)\Gamma(\nu)} \int_0^x (x-y)^{\mu-1} e^{-\alpha(x-y)} y^{\nu-1} e^{-\alpha y} dy \\
&= \frac{\alpha^{\mu+\nu}}{\Gamma(\mu)\Gamma(\nu)} e^{-\alpha x} \int_0^x (x-y)^{\mu-1} y^{\nu-1} dy \\
&= \frac{\alpha^{\mu+\nu}}{\Gamma(\mu)\Gamma(\nu)} e^{-\alpha x} \int_0^1 (x-xt)^{\mu-1} (xt)^{\nu-1} x dt \quad (\text{by } y = xt) \\
&= \frac{1}{\Gamma(\mu+\nu)} \alpha^{\mu+\nu} x^{\mu+\nu-1} e^{-\alpha x} \cdot \underbrace{\frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)} \int_0^1 (1-t)^{\mu-1} t^{\nu-1} dt}_{=1} \\
&= f_{\alpha,\mu+\nu}(x),
\end{aligned}
$$

where "that the last term is equal to one" follows the beta integral. □

For $\mu > 0$ and $\nu > 0$, $B(\mu,\nu) = \int_0^1 (1-t)^{\mu-1} t^{\nu-1} dt = \int_0^\infty \frac{t^{\mu-1}}{(1+t)^{\mu+\nu}} dt = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$ is the so-called *beta integral*.

Moment generating function

That the pdf of chi-square distribution with 1 degree of freedom equals $f_{1/2,1/2}(y)$ can be derived as:

$\Pr[Y_1^2 \le y] = \Pr[-\sqrt{y} \le Y_1 \le \sqrt{y}] = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\Phi(\sqrt{y}) - 1,$

where $\Phi(\cdot)$ represents the unit Gaussian cdf. So the derivative of the above equation is $y^{-1/2}\phi(y^{1/2}) = \dfrac{1}{\Gamma(1/2)}(1/2)^{1/2}y^{(1/2)-1}e^{-(1/2)y}$ for $y \ge 0$. Then by closure under convolution for gamma densities, the distribution of chi-square distribution with n degree of freedom can be obtained.

In statistical mechanics, $Y_1^2 + Y_2^2 + Y_3^2$ appears as the square of the speed of particles. So the density of particle speed (on $[0, \infty)$) is equal to $v(x) = 2x f_{1/2,3/2}(x^2)$, which is called *Maxwell density*.

By letting $\bar{Y} = \sqrt{Y_1^2 + \cdots + Y_n^2}$,

$$\begin{aligned} \Pr[T_n \le t] &= \Pr[(X\sqrt{n})/\bar{Y} \le t] = \int_0^\infty \Pr[X \le yt/\sqrt{n}]dF_{\bar{Y}}(y) \\ &= \int_0^\infty \Phi(yt/\sqrt{n})\big(2y\, f_{1/2,n/2}(y^2)\big)dy. \end{aligned}$$

Moment generating function

So the density of T_n is:

$$
\begin{aligned}
f_{T_n}(t) &= \int_0^\infty \left(\frac{y}{\sqrt{n}}\phi(yt/\sqrt{n})\right)\left(2y\, f_{1/2,n/2}(y^2)\right) dy \\
&= \int_0^\infty \left(\frac{y}{\sqrt{2\pi n}} e^{-y^2t^2/(2n)}\right)\left(\frac{1}{2^{n/2-1}\Gamma(n/2)} y^{n-1} e^{-y^2/2}\right) dy \\
&= \frac{1}{2^{(n-1)/2}\Gamma(n/2)\sqrt{\pi n}} \int_0^\infty y^n e^{-y^2(1+t^2/n)/2} dy \\
&= \frac{1}{2^{(n-1)/2}\Gamma(n/2)\sqrt{\pi n}} \int_0^\infty \frac{2^{n/2}s^{n/2}}{(1+t^2/n)^{n/2}} e^{-s}\left(\frac{2^{-1/2}s^{-1/2}}{(1+t^2/n)^{1/2}}\right) ds \quad \left(\text{by } y = \frac{\sqrt{2s}}{(1+t^2/n)^{1/2}}\right) \\
&= \frac{1}{\Gamma(n/2)\sqrt{\pi n}(1+t^2/n)^{(n+1)/2}} \int_0^\infty s^{(n+1)/2-1} e^{-s} ds \\
&= \frac{C_n}{(1+t^2/n)^{(n+1)/2}}, \text{ where } C_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}}.
\end{aligned}
$$

Moment generating function

For $t \neq 0$, the integral

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{tx} \frac{C_n}{(1+x^2/n)^{(n+1)/2}} dx &\geq C_n \int_0^{\infty} e^{|t|x} \frac{1}{(1+x^2/n)^{(n+1)/2}} dx \\
&\geq \frac{C_n}{n!} \int_0^{\infty} \frac{|t|^n x^n}{(1+x^2/n)^{(n+1)/2}} dx \\
&\quad \left(\text{by } e^x \geq 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \geq \frac{x^n}{n!} \text{ for } x \geq 0\right) \\
&\geq \frac{C_n}{n!} \int_{\sqrt{n}}^{\infty} \frac{|t|^n x^n}{(1+x^2/n)^{(n+1)/2}} dx \\
&\geq \frac{C_n}{n!} \int_{\sqrt{n}}^{\infty} \frac{|t|^n x^n}{(x^2/n+x^2/n)^{(n+1)/2}} dx = \frac{C_n |t|^n n^{(n+1)/2}}{n! 2^{(n+1)/2}} \int_{\sqrt{n}}^{\infty} \frac{1}{x} dx = \infty.
\end{aligned}$$

So the moment generating function is not defined for any $t \neq 0$.

Moment generating function

However for $r < n$,

$$\begin{aligned} E[|T_n|^r] &= \int_{-\infty}^{\infty} C_n \frac{|t|^r}{(1+t^2/n)^{(n+1)/2}} dt \\ &= 2C_n \int_0^{\infty} \frac{t^r}{(1+t^2/n)^{(n+1)/2}} dt = C_n n^{(r+1)/2} \int_0^{\infty} \frac{s^{(r+1)/2-1}}{(1+s)^{(r+1)/2+(n-r)/2}} ds \\ &= C_n n^{(r+1)/2} \frac{\Gamma((r+1)/2)\Gamma((n-r)/2)}{\Gamma((n+1)/2)} = n^{r/2} \frac{\Gamma((r+1)/2)\Gamma((n-r)/2)}{\Gamma(n/2)\sqrt{\pi}} < \infty. \end{aligned}$$

But when $r = n$ (similarly for $r > n$),

$$E[(T_n^r)^+] = E[(T_n^n)^+] = \left\{ \begin{array}{ll} \int_{-\infty}^{\infty} C_n \frac{t^n}{(1+t^2/n)^{(n+1)/2}} dt, & \text{if } n \text{ even;} \\ \int_0^{\infty} C_n \frac{t^n}{(1+t^2/n)^{(n+1)/2}} dt, & \text{if } n \text{ odd.} \end{array} \right\} = \infty$$

and

$$E[(T_n^r)^-] = E[(T_n^n)^-] = \begin{cases} 0, & \text{if } n \text{ even;} \\ \int_{-\infty}^{0} C_n \frac{-t^n}{(1+t^2/n)^{(n+1)/2}} dt = \infty, & \text{if } n \text{ odd.} \end{cases}$$

This is a good example for which the moments, even if some of them exist (and are finite), cannot be obtained through the moment generating function.

Some well-known densities

gamma distribution : $f_{\alpha,\nu}(x)$ (on $[0,\infty)$) has mean ν/α and variance ν/α^2.

Snedecor's distribution or F-distribution : It is the distribution of

$$Z = \frac{\dfrac{X_1^2 + \cdots + X_m^2}{m}}{\dfrac{Y_1^2 + \cdots + Y_n^2}{n}},$$

where X_i and Y_j are all independent standard normal. Its pdf with positive integer parameters m and n is:

$$\frac{m^{m/2}}{n^{m/2}} \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \frac{z^{m/2-1}}{(1+mz/n)^{(m+n)/2}} \quad \text{on } z \geq 0.$$

Bilateral exponential distribution : It is the distribution of $Z = X_1 - X_2$, where X_1 and X_2 are independent and have common exponential density $\alpha e^{-\alpha x}$ on $x \geq 0$.

Its pdf is $\dfrac{1}{2}\alpha e^{-\alpha|x|}$ on $x \in \Re$.

It has zero mean and variance $2\alpha^{-2}$.

Some well-known densities

Beta distribution : Its pdf with parameters $\mu > 0$ and $\nu > 0$ is

$$\beta_{\mu,\nu}(x) = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)}(1-x)^{\mu-1}x^{\nu-1} \quad \text{on } 0 < x < 1.$$

Its mean is $\nu/(\mu+\nu)$, and its variance is $\mu\nu/[(\mu+\nu)^2(\mu+\nu+1)]$.

Arc sine distribution : Its pdf is $\beta_{1/2,1/2}(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$ on $0 < x < 1$.

Its cdf is given by $\frac{2}{\pi}\sin^{-1}(\sqrt{x})$ for $0 < x < 1$.

Generalized arc sine distribution : It is a beta distribution with $\mu + \nu = 1$.

Pareto distribution : It is the distribution of $Z = X^{-1} - 1$, where X is beta distributed with parameters $\mu > 0$ and $\nu > 0$. Its density is

$$\frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)}\frac{z^{\mu-1}}{(1+z)^{\mu+\nu}} \quad \text{on } 0 < z < \infty.$$

This is often used as an incoming traffic with heavy tail as $z^{-(\nu+1)}$.

Some well-known densities

Cauchy distribution : It is the distribution of $Z = \alpha X/|Y|$, where X and Y are independent standard normal distributed.

Its pdf with parameter $\alpha > 0$ is

$$\gamma_\alpha(x) = \frac{1}{\pi}\frac{\alpha}{x^2+\alpha^2} \quad \text{on } \Re.$$

Its cdf is $\dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}(x/\alpha)$.

It is also **closure under convolution**, i.e., $\gamma_s * \gamma_u = \gamma_{s+u}$.

It is interesting that Cauchy distribution is also **closure under scaling**, i.e., $a \cdot X$ has density $\gamma_{a\alpha}(\cdot)$, if X has density $\gamma_\alpha(\cdot)$.
Hence, we can easily obtain the density of $a_1X_1 + a_2X_2 + \cdots + a_nX_n$ as $\gamma_{a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n}(\cdot)$, if X_i has density $\gamma_{\alpha_i}(\cdot)$.

Some well-known densities

One-sided stable distribution of index $1/2$: It is the distribution of $Z = \alpha^2 X^{-2}$, where X has a standard normal distribution.

Its pdf is $f_\alpha(z) = \dfrac{\alpha}{\sqrt{2\pi}} \dfrac{1}{\sqrt{z^3}} e^{-\frac{1}{2}\alpha^2/z}$ on $z \geq 0$.

It is also **closure under convolution**, namely, $f_\alpha * f_\beta = f_{\alpha+\beta}$.

If $Z_1, Z_2, \ldots, Z_n$ are i.i.d. with marginal density $f_\alpha(\cdot)$, then $\dfrac{Z_1 + \cdots + Z_n}{n^2}$ also has density $f_\alpha(\cdot)$.

Some well-known densities

Weibull distribution : Its pdf and cdf are respectively given by

$$\frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-(y/\beta)^{\alpha}} \quad \text{and} \quad 1 - e^{-(y/\beta)^{\alpha}}$$

with $\alpha > 0$, $\beta > 0$ and support $(0, \infty)$.

By defining $X = 1/Y$, where Y has the above distribution, we derive the pdf and cdf of X as respectively:

$$\alpha\beta(\beta y)^{-1-\alpha} e^{-1/(\beta y)} \quad \text{and} \quad e^{-1/(\beta x)^{\alpha}}.$$

This is useful for ordered statistics.

For example, let $X_{(n)}$ denote the largest one among i.i.d. $X_1, X_2, \ldots, X_n$.

Then $\Pr\left[\frac{X_{(n)}}{n} \leq x\right] = e^{-(\alpha/\pi)(1/x)}$, if X_j is Cauchy distributed with parameter α.

Or $\Pr\left[\frac{X_{(n)}}{n^2} \leq x\right] = e^{-(\alpha\sqrt{2}/\sqrt{\pi})(1/x^{1/2})}$, if X_j is a one-sided stable distribution of index $1/2$ with parameter α.

Logistic distribution : Its cdf with parameters $\alpha > 0$ and $\beta \in \Re$ is $\dfrac{1}{1 + e^{-\alpha x - \beta}}$ on $\Re$.

Distribution with nice moment generating function

Gaussian distribution :

$$M(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{-(x-m)^2/(2\sigma^2)} dx = e^{mt+\sigma^2 t^2/2},$$

which exists for all $t \in \Re$.

Exponential distribution :

$$M(t) = \int_0^{\infty} e^{tx} \alpha e^{-\alpha x} dx = \frac{\alpha}{\alpha - t} = \sum_{k=0}^{\infty} \left(\frac{k!}{\alpha^k} \right) \frac{t^k}{k!}$$

is defined for $t < \alpha$.

So the kth moment is $k! \alpha^{-k}$.

Poisson distribution :

$$M(t) = \sum_{r=0}^{\infty} e^{rt} e^{-\lambda} \frac{\lambda^r}{r!} = e^{\lambda(e^t - 1)},$$

which exists for all $t \in \Re$.