Order Statistics of Cumulative Sums

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Cumulative sum

Notations Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables.

Let $S_n = X_1 + \cdots + X_n$, and $S_0 = 0$.

Denote by $S_{(1)}, S_{(2)}, \ldots, S_{(n)}$ the order statistics of $S_1, S_2, \ldots, S_n$.

Invariance principle and extreme order statistics

Assumption $X_1, \ldots, X_n$ are i.i.d. with marginal mean 0 and marginal variance $\sigma^2 > 0$.

Theorem 37.7 (Skorohod embedding theorem) Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and finite variance.
Let $S_n = X_1 + \cdots + X_n$.
Then there is a non-decreasing sequence of stopping times $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots$ such that

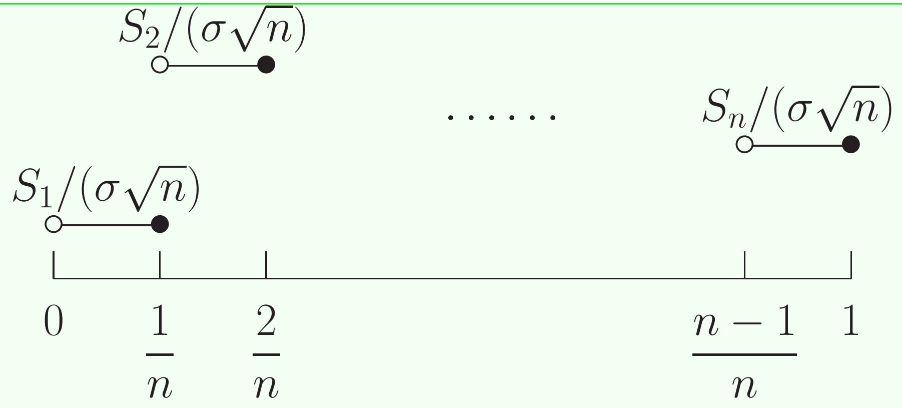
1. $W_{\boldsymbol{\tau}_n}$ (Brownian motion) has the same distribution as S_n, and

2. $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1, \boldsymbol{\tau}_3 - \boldsymbol{\tau}_2, \ldots$ are i.i.d. with

$$E[\boldsymbol{\tau}_n - \boldsymbol{\tau}_{n-1}] = E[X_1^2]$$

and

$$E[(\boldsymbol{\tau}_n - \boldsymbol{\tau}_{n-1})^2] \leq 4E[X_1^4].$$



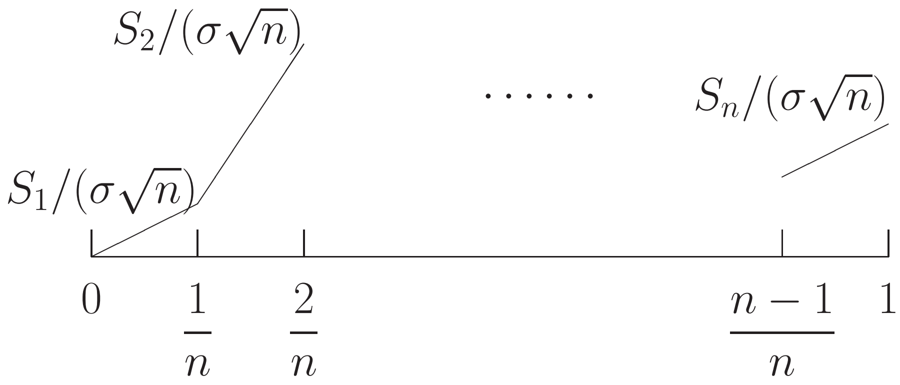
Define for each integer n a random process $\{Y_t(n), 0 \le t \le 1\}$ as:

$$Y_t(n) = \frac{S_{\lceil nt \rceil}}{\sigma\sqrt{n}}.$$

Theorem 37.8 if $E[X_1^4] < \infty$, there exist $\{Z_t(n), 0 \le t \le 1\}$ and $\{W_t(n), 0 \le t \le 1\}$ such that

1. $\{Z_t(n), 0 \le t \le 1\}$ and $\{Y_t(n), 0 \le t \le 1\}$ have the same **finite** dimensional distribution;
2. $\{W_t(n), 0 \le t \le 1\}$ is a Brownian motion;
3. $\lim_{n\to\infty} \Pr\left[\sup_{0\le t\le 1} |Z_t(n) - W_t(n)| \ge \varepsilon\right] = 0$. (We need to know the joint distribution between $Z_t(n)$ and $W_t(n)$ in order to evaluate the mass here.)

Invariance principle and extreme order statistics



By similar idea of invariance principle,

$$\bar{Y}_t(n) = \frac{S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)X_{\lfloor nt \rfloor + 1}}{\sigma\sqrt{n}} \Rightarrow W_t \text{ for } 0 \le t \le 1,$$

where W_t is a Wiener process. (This is a broken-line generated by the end-points of $(k/n, S_k/(\sigma\sqrt{n}))$.)

The above is useful in the following:

$$\Pr\left[\sup_{0\le t\le 1} \bar{Y}_t(n) \le x\right] \stackrel{n\to\infty}{\longrightarrow} \Pr\left[\sup_{0\le t\le 1} W_t \le x\right]$$

or

$$\Pr\left[\inf_{0\le t\le 1} \bar{Y}_t(n) \le x\right] \stackrel{n\to\infty}{\longrightarrow} \Pr\left[\inf_{0\le t\le 1} W_t \le x\right]$$

Invariance principle and extreme order statistics

or

$$\Pr\left[\sup_{0\le t\le 1}\bar{Y}_t(n)\le x\wedge\inf_{0\le t\le 1}\bar{Y}_t(n)\le y\wedge\bar{Y}_1(n)\le z\right]$$
$$\xrightarrow{n\to\infty}\ \Pr\left[\sup_{0\le t\le 1}W_t\le x\wedge\inf_{0\le t\le 1}W_t\le y\wedge W_1\le z\right].$$

Observe that

$$\frac{S_{(n)}}{\sigma\sqrt{n}}=\sup_{0\le t\le 1}\bar{Y}_t(n)\quad\text{and}\quad\frac{S_{(1)}}{\sigma\sqrt{n}}=\inf_{0\le t\le 1}\bar{Y}_t(n)\quad\text{and}\quad\frac{S_n}{\sigma\sqrt{n}}=\bar{Y}_1(n).$$

This immediately gives that:

$$\Pr\left[S_{(n)}\le x\sigma\sqrt{n}\wedge S_{(1)}\le y\sigma\sqrt{n}\wedge S_n\le z\sigma\sqrt{n}\right]\xrightarrow{n\to\infty}\Pr\left[\sup_{0\le t\le 1}W_t\le x\wedge\inf_{0\le t\le 1}W_t\le y\wedge W_1\le z\right].$$

How about independent but non-identically distributed variables?

Invariance principle and extreme order statistics

Let $\sigma_k^2 = \mathrm{Var}[X_k]$ and $E[X_k] = 0$.

Let $s_n^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$.

Then re-define the broken-line process $\bar{Y}_t(n)$ by points $(s_k^2/s_n^2, S_k/s_n)$.

In this case, Prohorov proved the invariance principle is also valid, i.e.,

$$\bar{Y}_n(t) \Rightarrow W_t \text{ for } 0 \le t \le 1,$$

if S_n/s_n converges to normal distribution.

Theorem If $X_1, X_2, \ldots, X_n$ are zero-mean independent variables, satisfying the Lindeberg condition, then

$$\Pr[S_{(n)} \le x s_n \wedge S_{(1)} \le y s_n \wedge S_n < z s_n] \stackrel{n\to\infty}{\longrightarrow} \Pr\left[\sup_{0\le t\le 1} W_t \le x \wedge \inf_{0\le t\le 1} W_t \le y \wedge W_1 \le z\right].$$

Can the theorem be further generalized?

Theorem If $X_1, X_2, \ldots, X_n$ are i.i.d., and S_n/s_n converges in distribution to a zero-mean random variable with characteristic function

$$\varphi(\theta) = \exp\left\{-\sigma^\alpha|\theta|^\alpha\left[1 + i\beta\cdot\text{sign}(\theta)\cdot\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\}$$

with $\sigma > 0$, $0 < \alpha \le 2$,

$$\begin{cases} |\beta| < 1, & \text{if } 0 < \alpha < 1; \\ \beta = 0, & \text{if } \alpha = 1; \\ |\beta| \le 1, & \text{if } 1 < \alpha \le 2, \end{cases} \quad \text{and} \quad \text{sign}(\theta) = \begin{cases} 1, & \text{if } \theta > 0; \\ 0, & \text{if } \theta = 0; \\ -1, & \text{if } \theta < 0, \end{cases}$$

then

$$\Pr\left[S_{(n)} \le xs_n \wedge S_{(n)} \le ys_n \wedge S_n \le zs_n\right] \stackrel{n\to\infty}{\longrightarrow} \Pr\left[\sup_{0\le t\le 1} Z_t \le x \wedge \inf_{0\le t\le 1} Z_t \le y \wedge Z_1 \le z\right],$$

where Z_t be a **stable process** with independent increments, where $Z_t - Z_s$ has characteristic function $\varphi^{t-s}(\theta)$, and $Z_0 = 0$.

Invariance principle and extreme order statistics

How to determine $\Pr\left[\sup_{0\le t\le 1} W_t \le x\right]$**?**

Answer: Fix a constant $x > 0$.
Since $\Pr[W_1 = x] = 0$,

$$\begin{aligned}\Pr\left[\sup_{0\le t\le 1} W_t \ge x\right] &= \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 \ge x\right] + \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 \le x\right] \\ &= \Pr\left[W_1 \ge x\right] + \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 \le x\right].\end{aligned}$$

Since (over the inherited probability space $(\Omega, \mathcal{F}, P)$) path $W_t(\omega)$ is continuous in t, there exists $\tau(\omega)$ such that

$$\left\{\omega \in \Omega \;:\; \sup_{0\le t\le 1} W_t(\omega) \ge x\right\} = \left\{\omega \in \Omega \;:\; W_{\tau(\omega)}(\omega) = x\right\}.$$

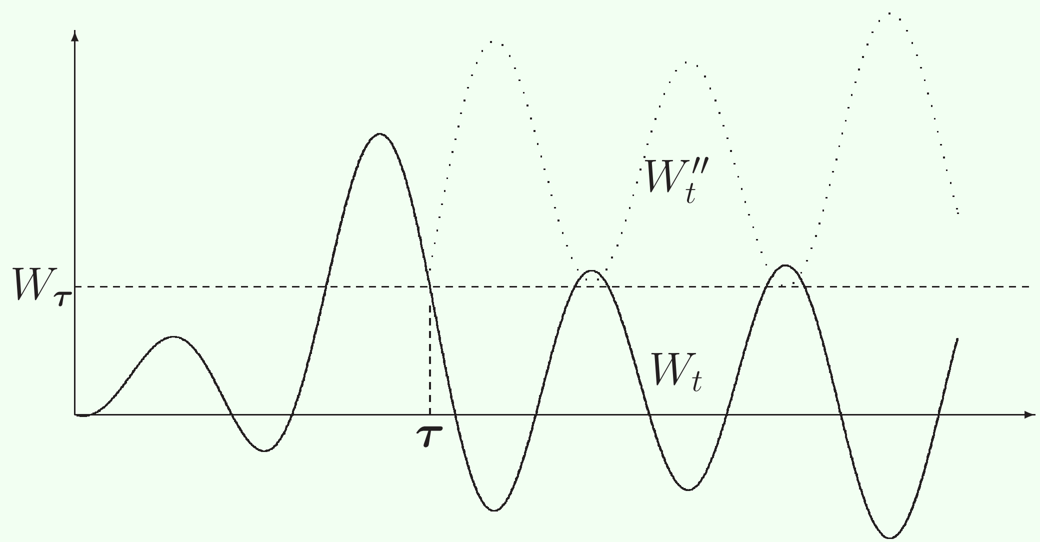
Therefore, $\boldsymbol{\tau}$ is a random variable defined over $(\Omega, \mathcal{F}, P)$ such that the two events below are equivalent:

$$\left[\sup_{0\le t\le 1} W_t \ge x\right] = \left[W_{\boldsymbol{\tau}} = x\right].$$

The reflection principle.

For a stopping time $\boldsymbol{\tau}$ (non-negative random variable), define

$$W''_t = \begin{cases} W_t, & \text{if } t \le \boldsymbol{\tau}; \\ W_{\boldsymbol{\tau}} - (W_t - W_{\boldsymbol{\tau}}), & \text{if } t \ge \boldsymbol{\tau}. \end{cases}$$



As anticipated, W''_t is a Brownian motion.

Invariance principle and extreme order statistics

By the reflection principle,

$$\begin{aligned}
\Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 \le x\right] &= \Pr\left[W_{\boldsymbol{\tau}} = x \wedge W_1 \le x\right] \\
&= \Pr\left[W''_{\boldsymbol{\tau}} = x \wedge W''_1 \le x\right] \\
&= \Pr\left[W_{\boldsymbol{\tau}} = x \wedge 2W_{\boldsymbol{\tau}} - W_1 \le x\right] \\
&\quad \text{(Take } t = \boldsymbol{\tau} \text{ for the 1st term, and } t = 1 \text{ for the 2nd term.)} \\
&= \Pr\left[W_{\boldsymbol{\tau}} = x \wedge W_1 \ge x\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 \ge x\right] \\
&= \Pr\left[W_1 \ge x\right],
\end{aligned}$$

which implies that for $x > 0$:

$$\Pr\left[\sup_{0\le t\le 1} W_t \ge x\right] = \begin{cases} 2\Pr\left[W_1 \ge x\right] = 2(1 - \Phi(x)), & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\Phi(\cdot)$ is the unit Gaussian cdf. We conclude that for real x,

$$\Pr\left[\sup_{0\le t\le 1} W_t \le x\right] = \begin{cases} 2\Phi(x) - 1, & x > 0 \\ 0, & \text{otherwise} \end{cases} = \max\{0, 2\Phi(x) - 1\}$$

How to determine $\Pr\left[\sup_{0\le t\le 1} W_t \le x \wedge W_1 \le y\right]$**?**

Answer: Again, use the reflection principle.

$$\begin{aligned}
\Phi(y) &= \Pr[W_1 < y] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] + \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 < y\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] + \Pr\left[W_{\boldsymbol{\tau}} = x \wedge W_1 < y\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] + \Pr\left[W''_{\boldsymbol{\tau}} = x \wedge W''_1 < y\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] + \Pr\left[W_{\boldsymbol{\tau}} = x \wedge 2W_{\boldsymbol{\tau}} - W_1 < y\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] + \Pr\left[W_{\boldsymbol{\tau}} = x \wedge W_1 > 2x - y\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] + \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 > 2x - y\right].
\end{aligned}$$

Consequently,

$$\Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] = \begin{cases} \Pr\left[\sup_{0\le t\le 1} W_t < x\right], & \text{if } y \ge x > 0; \\ \Phi(y) - \Pr\left[\sup_{0\le t\le 1} W_t \ge x \wedge W_1 > 2x - y\right], & \text{if } y < x \\ & \text{(i.e., } 2x - y > x) \end{cases}$$

$$= \begin{cases} 2\Phi(x) - 1, & \text{if } y \ge x > 0; \\ \Phi(y) - \Pr\left[W_1 > 2x - y\right], & \text{if } y < x; \quad (2x - y > x) \end{cases}$$

$$= \begin{cases} 2\Phi(x) - 1, & \text{if } y \ge x > 0; \\ \Phi(y) + \Phi(2x - y) - 1, & \text{if } y < x. \end{cases}$$

How to determine the cdf of $\inf_{0\le t\le 1} W_t$?

Using the reflection principle with $\tau = 0$, we have

$$\begin{aligned}
\Pr\left[\inf_{0\le t\le 1} W_t < x\right] &= \Pr\left[\inf_{0\le t\le 1} W''_t < x\right] \\
&= \Pr\left[\inf_{0\le t\le 1} (-W_t) < x\right] \\
&= \Pr\left[\sup_{0\le t\le 1} W_t > -x\right] \\
&= 1 - \max\{0, 2\Phi(-x) - 1\} \\
&= \min\{1, 2\Phi(x)\}.
\end{aligned}$$

Invariance principle and extreme order statistics

How to determine the cdf of $\sup_{0\le t\le 1}|W_t|$**?**

Also by repeatedly using the reflection principle (details are omitted here),

$$\begin{aligned}\Pr\left[\sup_{0\le t\le 1}|W_t|<x\right] &= \Pr\left[-x<\inf_{0\le t\le 1}W_t\le \sup_{0\le t\le 1}W_t<x\right]\\ &= \sum_{k=-\infty}^{\infty}(-1)^k\big[\Phi((2k+1)x)-\Phi((2k-1)x)\big]\\ &= 1-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\exp\left\{-\frac{\pi^2(2k+1)^k}{8x^2}\right\}.\end{aligned}$$

Brownian bridge

When the concerned distribution of **maximal order statistics** is the sum sequence $S_1, S_2, \ldots, S_n$ given that $S_n = g$, a Brownian bridge becomes the limit process instead of the Brownian motion.

Definition A Brownian bridge $\{W_t^{(a)}, 0 \leq t \leq 1\}$ is a Wiener process W_t conditioned on $W_1 = a$.

Brownian bridge

$$\Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < y\right] = \begin{cases} 2\Phi(x)-1, & \text{if } y \ge x > 0; \\ \Phi(y)+\Phi(2x-y)-1, & \text{if } y < x. \end{cases}$$

Distribution of $\sup_{0\le t\le 1} W_t^{(a)}$.

$$\begin{aligned}
&\Pr\left[\sup_{0\le t\le 1} W_t < x \,\middle|\, a \le W_1 < a+\varepsilon\right] \\
= &\frac{\Pr\left[\sup_{0\le t\le 1} W_t < x \wedge a \le W_1 < a+\varepsilon\right]}{\Pr\left[a \le W_1 < a+\varepsilon\right]} \\
= &\frac{\Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < a+\varepsilon\right] - \Pr\left[\sup_{0\le t\le 1} W_t < x \wedge W_1 < a\right]}{\Pr\left[a \le W_1 < a+\varepsilon\right]} \\
= &\begin{cases} 0, & \text{if } x \le a; \\ \dfrac{2\Phi(x)-\Phi(a)-\Phi(2x-a)}{\Phi(a+\varepsilon)-\Phi(a)}, & \text{if } a < x \le a+\varepsilon; \\ \dfrac{\Phi(a+\varepsilon)+\Phi(2x-a-\varepsilon)-\Phi(a)-\Phi(2x-a)}{\Phi(a+\varepsilon)-\Phi(a)}, & \text{if } x > a+\varepsilon. \end{cases}
\end{aligned}$$

Hence, as $\varepsilon \downarrow 0$,

$$\Pr\left[\sup_{0\le t\le 1} W_t^{(a)} < x\right] = 1 - e^{-2x(x-a)} \text{ for } x > a.$$