Berry-Esseen Theorem

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Historical aspects

- The central limit theorem (CLT) concerns the situation that the limit distribution of the normalized sum is normal.

- As an example, for i.i.d. zero-mean sequence $X_1, X_2, \ldots$,

$$\frac{X_1 + \cdots + X_n}{\sqrt{nE[X_1^2]}} \Rightarrow N,$$

  where N has standard normal distribution.

- **Question:** What is the rate of convergence of normalized sum distribution to standard normal distribution?

Historical aspects

- The first convergence rate estimates in the CLT were obtained by A. M. Lyapounov in 1900-1901.
- In the beginning of 1940s, the classic Berry-Esseen estimate came to the light:

$$\sup_{x\in\Re} |F_n(x) - \Phi(x)| \leq C \frac{\beta_3}{\sigma^3\sqrt{n}},$$

  where

  - F_n is the cdf of $(X_1 + \cdots + X_n)/\sqrt{nE[X_1^2]}$,
  - Φ is the standard normal cdf,
  - $\beta_3 = E\left[|X - E[X]|^3\right]$,
  - $\sigma^2 = E\left[|X - E[X]|^2\right]$, and
  - C is a universal constant, independent of n.

- In fact, it was due to Lyapounov that the finiteness of $(2 + \delta)$th absolute moment, where $0 < \delta < 1$, guarantees

$$\sup_{x\in\Re} |F_n(x) - \Phi(x)| = O(n^{-\delta/2}) \text{ as } n \to \infty.$$

Berry-Esseen theorem

- Classic Berry-Esseen theorem
  - For independent zero-mean random variables $\{X_i\}_{i=1}^n$,

$$\sup_{a\in\Re}\left|\Pr\left[\frac{1}{s_n}(X_1+\cdots+X_n)\le a\right]-\Phi(a)\right|\le C\,\frac{r_n}{s_n^3},$$

    where

    * $s_n^2 \equiv$ sum of the marginal variances,
    * $r_n \equiv$ sum of the marginal absolute 3rd central moments,
    * $C \equiv$ absolute constant, and
    * $\Phi(\cdot) \equiv$ the unit Gaussian cumulative distribution function (cdf).
  - Notably, only the *first three moments* are involved in the Berry-Esseen inequality.

Berry-Esseen theorem

$$\sup_{a\in\Re}\left|\Pr\left[\frac{1}{s_n}(X_1+\cdots+X_n)\le a\right]-\Phi(a)\right|\le C\,\frac{r_n}{s_n^3}.$$

- (Feller's book, 1979)
  - $C = 6$ for an independent sample sum.
  - $C = 3$ for an independent and identically distributed sample sum.
- (Beek 1972) $C = 0.7975$ for an independent sample sum.
- (Shiganov 1986)
  - $C = 0.7915$ for an independent sample sum.
  - $C = 0.7655$ for an independent and identically distributed sample sum.

Berry-Esseen theorem

- Again, the remarkable aspect of the Berry-Esseen theorem is that the upper bound depends only on the variance and the 3rd central moment.
- Hence, it can provide a good probability estimate based on the first three moments.

Berry-Esseen theorem and probability bound

Technique (behind the proof): Pass the difference through a bandlimited filter.

Lemma Fix a symmetric bandlimited filtering function

$$v_{T}(x) = \frac{1-\cos(Tx)}{\pi T x^2} = \frac{2\sin^2(Tx/2)}{\pi T x^2}$$

with characteristic function

$$\omega_{T}(\zeta) \triangleq \int_{-\infty}^{\infty} v_{T}(x) e^{-j\zeta x} dx = \begin{cases} 1 - \frac{|\zeta|}{T}, & \text{if } |\zeta| \le T; \\ 0, & \text{otherwise.} \end{cases}$$

For any cdf $H(\cdot)$ on the real line $\Re$,

$$\sup_{x\in\Re} |\Delta_T(x)| \ge \frac{1}{2}\eta - \frac{6}{T\pi\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2}\eta\right),$$

where $\eta \triangleq \sup_{x\in\Re} |H(x) - \Phi(x)|$, $\Delta_T(t) \triangleq \int_{-\infty}^{\infty} [H(t-x) - \Phi(t-x)] \times v_{T}(x) dx$,
and $h(u) \triangleq \pi u \int_{u}^{\infty} v_{T}\left(\frac{t}{T}\right) \frac{dt}{T} = \frac{\pi}{2} u + 1 - \cos(u) - u \int_{0}^{u} \frac{\sin(x)}{x} dx.$

Berry-Esseen theorem and probability bound

Summary: η is the desired difference between any cdf $H(\cdot)$ and standard normal cdf $\Phi(\cdot)$.

$$\begin{aligned}
\sup_{x\in\Re}\left|[H(x)-\Phi(x)]*v_T(x)\right| \\
&\geq \frac{1}{2}\eta - \frac{6}{T\pi\sqrt{2\pi}}\cdot\pi\cdot\frac{T\sqrt{2\pi}}{2}\eta\int_{T\sqrt{2\pi}\eta/2}^{\infty} v_T\left(\frac{t}{T}\right)\frac{dt}{T} \\
&= \frac{1}{2}\eta - 3\eta\int_{\sqrt{2\pi}\eta/2}^{\infty} v_T(y)dy \\
&= \eta\left(\frac{1}{2} - 3\int_{\sqrt{2\pi}\eta/2}^{\infty} v_T(y)dy\right) \\
&= \left(\sup_{x\in\Re}|H(x)-\Phi(x)|\right)\left(\frac{1}{2} - 3\int_{\sqrt{2\pi}\eta/2}^{\infty} v_T(y)dy\right)
\end{aligned}$$

Observation

- *The maximum absolute value of filter output* bounds the *maximum absolute value of filter input.*

Berry-Esseen theorem and probability bound

Proof:

- The right-continuity of the cdf $H(\cdot)$ and the continuity of the Gaussian unit cdf $\Phi(\cdot)$ together indicate the right-continuity of $|H(x) - \Phi(x)|$, which in turn implies the existence of $x_0 \in \Re$ satisfying

  $$\text{either} \quad \eta = |H(x_0) - \Phi(x_0)| \quad \text{or} \quad \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \,.$$

- We then distinguish between three cases:

$$\begin{array}{ll} \text{Case A)} & \eta = H(x_0) - \Phi(x_0); \\ \text{Case B)} & \eta = \Phi(x_0) - H(x_0); \\ \text{Case C)} & \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \,. \end{array}$$

Case A) $\eta = H(x_0) - \Phi(x_0)$.

In this case, we note that for $s > 0$,

$$\begin{aligned} H(x_0 + s) - \Phi(x_0 + s) &\geq H(x_0) - \left[\Phi(x_0) + \frac{s}{\sqrt{2\pi}}\right] && (1) \\ &= \eta - \frac{s}{\sqrt{2\pi}}, && (2) \end{aligned}$$

where (1) follows from $\sup_{x \in \Re} |\Phi'(x)| = 1/\sqrt{2\pi}$.

Observe that (2) implies

$$\begin{aligned} H\Bigg(x_0 + \underbrace{\frac{\sqrt{2\pi}}{2}\eta - x}_{=s}\Bigg) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right) &\geq \eta - \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{2\pi}}{2}\eta - x\right) \\ &= \frac{1}{2}\eta + \frac{x}{\sqrt{2\pi}}, \end{aligned}$$

for $|x| < \eta\sqrt{2\pi}/2$.

Together with the fact that $H(x) - \Phi(x) \geq -\eta$ for all $x \in \Re$, we obtain

$$\begin{aligned}
\sup_{x\in\Re} |\Delta_T(x)| &\geq \Delta_T\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta\right) \\
&= \int_{-\infty}^{\infty}\left[H\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right)\right] \times v_{_T}(x)dx \\
&= \int_{\left[|x|<\eta\sqrt{2\pi}/2\right]}\left[H\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right)\right] \times v_{_T}(x)dx \\
&+ \int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]}\left[H\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{2}\eta - x\right)\right] \times v_{_T}(x)dx \\
&\geq \int_{\left[|x|<\eta\sqrt{2\pi}/2\right]}\left[\frac{1}{2}\eta + \frac{x}{\sqrt{2\pi}}\right] \times v_{_T}(x)dx + \int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]}(-\eta) \times v_{_T}(x)dx \\
&= \int_{\left[|x|<\eta\sqrt{2\pi}/2\right]}\frac{1}{2}\eta \times v_{_T}(x)dx + \int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]}(-\eta) \times v_{_T}(x)dx, \qquad (3)
\end{aligned}$$

where the last equality holds because of the symmetry of the filtering function $v_{_T}(\cdot)$.

Berry-Esseen theorem and probability bound

The quantity of $\int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]} v_T(x)dx$ can be derived as follows:

$$\begin{aligned}
\int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]} v_T(x)dx &= 2\int_{\eta\sqrt{2\pi}/2}^{\infty} v_T(x)dx \\
&= 2\int_{\eta T\sqrt{2\pi}/2}^{\infty} v_T\left(\frac{u}{T}\right)\frac{du}{T} \\
&= \frac{4}{\eta T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{2}\eta\right).
\end{aligned}$$

Continuing from (3),

$$\begin{aligned}
\sup_{x\in\Re}|\Delta_T(x)| &\geq \frac{1}{2}\eta\left[1-\frac{4}{\eta T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{2}\eta\right)\right] \\
&\quad -\eta\cdot\left[\frac{4}{\eta T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{2}\eta\right)\right] \\
&= \frac{1}{2}\eta-\frac{6}{T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{2}\eta\right).
\end{aligned}$$

Berry-Esseen theorem and probability bound

Case B) $\eta = \Phi(x_0) - H(x_0)$.

Similar to Case A), we first derive for $s > 0$,

$$\Phi(x_0 - s) - H(x_0 - s) \geq \left[\Phi(x_0) - \frac{s}{\sqrt{2\pi}}\right] - H(x_0) = \eta - \frac{s}{\sqrt{2\pi}},$$

and then obtain

$$\begin{aligned}\Phi\left(x_0 - \frac{\sqrt{2\pi}}{2}\eta - x\right) - H\left(x_0 - \frac{\sqrt{2\pi}}{2}\eta - x\right) &\geq \eta - \frac{1}{\sqrt{2\pi}}\Big(\underbrace{\frac{\sqrt{2\pi}}{2}\eta + x}_{=s}\Big) \\ &= \frac{1}{2}\eta - \frac{x}{\sqrt{2\pi}},\end{aligned}$$

for $|x| < \eta\sqrt{2\pi}/2$.

Berry-Esseen theorem and probability bound

Together with the fact that $\Phi(x) - H(x) \geq -\eta$ for all $x \in \Re$, we obtain

$$\begin{aligned}
\sup_{x\in\Re} |\Delta_T(x)| &\geq -\Delta_T\left(x_0 - \frac{\sqrt{2\pi}}{2}\,\eta\right) \\
&\geq \int_{\left[|x|<\eta\sqrt{2\pi}/2\right]} \left[\frac{1}{2}\eta - \frac{x}{\sqrt{2\pi}}\right] \times v_{_T}(x)dx \\
&\quad + \int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]} (-\eta) \times v_{_T}(x)dx \\
&= \int_{\left[|x|<\eta\sqrt{2\pi}/2\right]} \frac{1}{2}\eta \times v_{_T}(x)dx + \int_{\left[|x|\geq\eta\sqrt{2\pi}/2\right]} (-\eta) \times v_{_T}(x)dx \\
&= \frac{1}{2}\,\eta - \frac{6}{T\pi\sqrt{2\pi}}\,h\left(\frac{T\sqrt{2\pi}}{2}\,\eta\right).
\end{aligned}$$

Berry-Esseen theorem and probability bound

Case C) $\eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \geq 0$.

In this case, we observe that for any $0 < \delta < \eta$, there exists x_0' such that $|H(x_0') - \Phi(x_0')| \geq \eta - \delta \triangleq \eta'$. We can then follow the procedure of the previous two cases to obtain:

$$\sup_{x \in \Re} |\Delta_T(x)| \geq \frac{1}{2}\eta' - \frac{6}{T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{2}\eta'\right).$$

The proof is completed by noting that η' can be made arbitrarily close to η. □

Berry-Esseen theorem and probability bound

Lemma For any cumulative distribution function $H(\cdot)$ with zero-mean and unit variance, its characteristic function $\varphi_H(\zeta)$ satisfies that

$$\eta \le \frac{1}{\pi}\int_{-T}^{T}\left|\varphi_H(\zeta) - e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|} + \frac{12}{T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}\,\eta}{2}\right),$$

where η and $h(\cdot)$ are defined in the previous lemma.

This lemma transforms the bound in the previous lemma into *frequency domain*, namely the *characteristic function domain*.

Berry-Esseen theorem and probability bound

Proof:

- Observe that

$$\Delta_T(t) = \int_{-\infty}^{\infty} [H(t-x) - \Phi(t-x)] \times v_{T}(x)dx$$

is nothing but a convolution of $v_{T}(\cdot)$ and $H(\cdot) - \Phi(\cdot)$.

- By Fourier inversion theorem,

$$\left.\frac{d\left(\Delta_T(t)\right)}{dt}\right|_{t=x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\zeta x}\left[\varphi_{H}(\zeta) - e^{-(1/2)\zeta^2}\right]\omega_{T}(\zeta)d\zeta$$
$$= \frac{1}{2\pi}\int_{-T}^{T} e^{-j\zeta x}\left[\varphi_{H}(\zeta) - e^{-(1/2)\zeta^2}\right]\omega_{T}(\zeta)d\zeta,$$

where the second step follows from bandlimit assumption.

Notably, $\varphi_H(\zeta)$ and $e^{-\zeta^2/2}$ are the Fourier transforms with respect to measures $dH(x)$ and $d\Phi(x)$, not the Fourier transforms of $H(x)$ and $\Phi(x)$.

Actually, $H(x)$ and $\Phi(x)$ have no Fourier transforms.

Berry-Esseen theorem and probability bound

- Integrating with respect to x, we obtain

$$\Delta_T(x) = \frac{1}{2\pi}\int_{-T}^{T} e^{-j\zeta x}\frac{\left[\varphi_H(\zeta) - e^{-(1/2)\zeta^2}\right]}{-j\zeta}\omega_T(\zeta)d\zeta, \tag{4}$$

where no integration constant appears since both sides go to zero as $|x| \to \infty$.

Notably,

$$f'(x) = g'(x) \Rightarrow f(x) = g(x) + c,$$

where c is the integration constant.

Accordingly,

$$
\begin{aligned}
\sup_{x\in\Re}|\Delta_T(x)| &= \sup_{x\in\Re}\frac{1}{2\pi}\left|\int_{-T}^{T} e^{-j\zeta x}\frac{\left[\varphi_H(\zeta)-e^{-(1/2)\zeta^2}\right]}{-j\zeta}\omega_T(\zeta)d\zeta\right| \\
&\le \sup_{x\in\Re}\frac{1}{2\pi}\int_{-T}^{T}\left|e^{-j\zeta x}\frac{\left[\varphi_H(\zeta)-e^{-(1/2)\zeta^2}\right]}{-j\zeta}\omega_T(\zeta)\right|d\zeta \\
&= \sup_{x\in\Re}\frac{1}{2\pi}\int_{-T}^{T}\left|\varphi_H(\zeta)-e^{-(1/2)\zeta^2}\right|\cdot|\omega_T(\zeta)|\frac{d\zeta}{|\zeta|} \\
&\le \sup_{x\in\Re}\frac{1}{2\pi}\int_{-T}^{T}\left|\varphi_H(\zeta)-e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|} \\
&= \frac{1}{2\pi}\int_{-T}^{T}\left|\varphi_H(\zeta)-e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|}.
\end{aligned}
$$

Berry-Esseen theorem and probability bound

Together with

$$\sup_{x\in\Re} |\Delta_T(x)| \geq \frac{1}{2}\,\eta - \frac{6}{T\pi\sqrt{2\pi}}\; h\left(\frac{T\sqrt{2\pi}}{2}\,\eta\right),$$

we finally have

$$\eta \;\leq\; \frac{1}{\pi}\int_{-T}^{T}\left|\varphi_H(\zeta) - e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|} + \frac{12}{T\pi\sqrt{2\pi}}\,h\left(\frac{T\sqrt{2\pi}}{2}\,\eta\right).$$

□

Berry-Esseen theorem and probability bound

Theorem Let $Y_n = \sum_{i=1}^{n} X_i$ be sum of i.i.d. random variables. Assume $n \geq 3$. Denote the mean and variance of X_1 by $\hat{\mu}$ and $\hat{\sigma}^2$, respectively.
Define

$$\hat{\rho} \triangleq E\left[|X_1 - \hat{\mu}|^3\right].$$

Also denote the cdf of $(Y_n - E[Y_n])/(\hat{\sigma}\sqrt{n})$ by $H_n(\cdot)$.
Then

$$\sup_{y \in \Re} |H_n(y) - \Phi(y)| \leq 5 \, \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}}.$$

Proof:

- $$\pi \cdot \eta \leq \int_{-T}^{T} \left| \hat{\varphi}^n \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - e^{-\zeta^2/2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta \right), \qquad (5)$$

  where $\hat{\varphi}(\cdot)$ is the characteristic function of $(X_1 - \hat{\mu})$.

Lemma For any complex numbers α and β,

$$|\alpha^n - \beta^n| \leq n|\alpha - \beta|\gamma^{n-1},$$

where $\gamma \geq \max\{|\alpha|, |\beta|\}$.

Proof:

$$\begin{aligned} |\alpha^n - \beta^n| &= \left| \left(\frac{\alpha}{r}\right)^n - \left(\frac{\beta}{r}\right)^n \right| r^n \\ &\leq n \left| \frac{\alpha}{r} - \frac{\beta}{r} \right| r^n \\ &= n|\alpha - \beta| r^{n-1}. \end{aligned}$$

∎

Berry-Esseen theorem and probability bound

- Observe that the integrand satisfies

$$\begin{aligned}
&\left|\hat{\varphi}^n\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right) - e^{-\zeta^2/2}\right| \\
&\quad \le\ n\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right) - e^{-\zeta^2/(2n)}\right|\gamma^{n-1}, \qquad (6)\\
&\quad \le\ n\left(\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right) - \left(1-\frac{\zeta^2}{2n}\right)\right| + \left|\left(1-\frac{\zeta^2}{2n}\right) - e^{-\zeta^2/(2n)}\right|\right)\gamma^{n-1} \qquad (7)
\end{aligned}$$

where the quantity γ in (6) requires that

$$\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right)\right| \le \gamma \quad \text{and} \quad \left|e^{-\zeta^2/(2n)}\right| \le \gamma.$$

We upperbound the first and second terms in the parentheses of (7) respectively by

$$\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right) - 1 + \frac{\zeta^2}{2n}\right| \le \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta|^3 \quad \text{and} \quad \left|1-\frac{\zeta^2}{2n} - e^{-\zeta^2/(2n)}\right| \le \frac{1}{8n^2}\zeta^4.$$

Continuing the derivation of (7),

$$\left|\hat{\varphi}^n\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right) - e^{-\zeta^2/2}\right| \ \le\ n\left(\frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta|^3 + \frac{1}{8n^2}\zeta^4\right)\gamma^{n-1}. \qquad (8)$$

It remains to choose γ that bounds both $|\hat{\varphi}(\zeta/(\hat{\sigma}\sqrt{n}))|$ and $\exp\{-\zeta^2/(2n)\}$ from above.

Berry-Esseen theorem and probability bound

For complex number z and reals b and c,

$$|z-b| \le c \Rightarrow |z| \le |b| + c.$$

Accordingly,

$$\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right) - 1 + \frac{\zeta^2}{2n}\right| \le \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta|^3 \Rightarrow \left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right)\right| \le \left|1 - \frac{\zeta^2}{2n}\right| + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta|^3.$$

- Next,

$$\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right)\right| \le 1 - \frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta^3|, \text{ if } \frac{\zeta^2}{2n} \le 1. \tag{9}$$

For those $\zeta \in [-T, T]$ (which is exactly the range of integration operation in (5)), we can guarantee the validity of the condition in (9) by defining

$$T \triangleq \frac{\hat{\sigma}^3\sqrt{n}}{\hat{\rho}}\left(\frac{\sqrt{2}n-3}{n-1}\right),$$

and obtain

$$\frac{\zeta^2}{2n} \le \frac{T^2}{2n} = \frac{\hat{\sigma}^6}{2\hat{\rho}^2}\left(\frac{\sqrt{2}n-3}{n-1}\right)^2 \le \frac{1}{2}\left(\frac{\sqrt{2}n-3}{n-1}\right)^2 \le 1,$$

for $n \ge 3$.

Berry-Esseen theorem and probability bound

Hence, for $|\zeta| \le T$,

$$\begin{aligned}
\left|\hat{\varphi}\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right)\right| &\le 1 + \left(-\frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta^3|\right) \\
&\le \exp\left\{-\frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}|\zeta^3|\right\} \\
&\le \exp\left\{-\frac{1}{2n}\zeta^2 + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}T\zeta^2\right\} \\
&= \exp\left\{-\left(\frac{1}{2n} - \frac{\hat{\rho}T}{6\hat{\sigma}^3 n^{3/2}}\right)\zeta^2\right\} \\
&= \exp\left\{-\frac{(3-\sqrt{2})}{6(n-1)}\zeta^2\right\}.
\end{aligned}$$

We can then choose

$$\gamma \triangleq \exp\left\{-\frac{(3-\sqrt{2})}{6(n-1)}\zeta^2\right\}.$$

Note that the above selected γ is an upper bound of $\exp\left\{-\zeta^2/(2n)\right\}$ for $n \ge 3/\sqrt{2} \approx 2.12$.

Berry-Esseen theorem and probability bound

- By taking the chosen γ into (8), the integration part in (5) becomes

$$
\begin{aligned}
&\int_{-T}^{T}\left|\hat{\varphi}^n\left(\frac{\zeta}{\hat{\sigma}\sqrt{n}}\right)-e^{-\zeta^2/2}\right|\frac{d\zeta}{|\zeta|}\\
\le\ &\int_{-T}^{T} n\left(\frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}}\zeta^2+\frac{1}{8n^2}|\zeta|^3\right)\cdot\exp\left\{-\frac{(3-\sqrt{2})}{6}\zeta^2\right\}d\zeta\\
\le\ &\int_{-\infty}^{\infty}\left(\frac{\hat{\rho}}{6\hat{\sigma}^3\sqrt{n}}\zeta^2+\frac{1}{8n}|\zeta|^3\right)\cdot\exp\left\{-\frac{(3-\sqrt{2})}{6}\zeta^2\right\}d\zeta\\
=\ &\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}}\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}}+\frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{\hat{\sigma}^3}{\hat{\rho}\sqrt{n}}\right)\\
\le\ &\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}}\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}}+\frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{1}{\sqrt{n}}\right)\\
=\ &\frac{1}{T}\left(\frac{\sqrt{2}n-3}{n-1}\right)\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}}+\frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{1}{\sqrt{n}}\right),
\end{aligned}
\tag{10}
$$

where the last inequality follows from Lyapounov's inequality, i.e.,

$$\hat{\sigma}=E^{1/2}\left[|X_{d+1}-\hat{\mu}|^2\right]\le E^{1/3}\left[|X_{d+1}-\hat{\mu}|^3\right]=\hat{\rho}^{1/3}.$$

Berry-Esseen theorem and probability bound

- Taking (10) into (5), we finally obtain

$$\pi \cdot \eta \;\leq\; \frac{1}{T}\left(\frac{\sqrt{2}n-3}{n-1}\right)\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}}+\frac{9}{2\left(3-\sqrt{2}\right)^{2}}\frac{1}{\sqrt{n}}\right)$$

$$+\frac{12}{T\sqrt{2\pi}}\,h\left(\frac{T\sqrt{2\pi}}{2}\,\eta\right),$$

or equivalently,

$$\pi u - 6h(u) \;\leq\; \frac{\sqrt{\pi}(2n-3\sqrt{2})}{4(n-1)}\left(\frac{\sqrt{6\pi}}{\left(3-\sqrt{2}\right)^{3/2}}+\frac{9}{\left(3-\sqrt{2}\right)^{2}}\frac{1}{\sqrt{n}}\right), \tag{11}$$

for $u \triangleq T\sqrt{2\pi}\eta/2$.

Berry-Esseen theorem and probability bound

Observe that the LHS of (11):

$$\pi u - 6h(u) = \pi u \left(1 - 6\int_u^{\infty} v_T\left(\frac{t}{T}\right)\frac{dt}{T}\right)$$

is continuous, and equals 0 at $u = 0$, and goes to ∞ as $u \to \infty$, which guarantees the existence of positive u satisfying (11).

Inequality (11) thus implies

$$u \;\le\; \max\left\{a \ge 0 \;:\; \pi a - 6h(a) \le \frac{\sqrt{\pi}(2n - 3\sqrt{2})}{4(n-1)}\left(\frac{\sqrt{6\pi}}{(3-\sqrt{2})^{3/2}} + \frac{9}{(3-\sqrt{2})^2}\frac{1}{\sqrt{n}}\right)\right\}.$$

Since

$$\frac{\sqrt{\pi}(2n - 3\sqrt{2})}{4(n-1)} = \frac{\sqrt{\pi}}{4}\left(2 - \frac{3\sqrt{2} - 2}{n-1}\right) \le \frac{\sqrt{\pi}}{2},$$

we can, for $n \ge 3$, further upper-bound u by:

$$\begin{aligned} u \;&\le\; \max\left\{a \ge 0 \;:\; \pi a - 6h(a) \le \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{6\pi}}{(3-\sqrt{2})^{3/2}} + \frac{9}{(3-\sqrt{2})^2}\frac{1}{\sqrt{3}}\right)\right\} \\ &= 3.26369\ldots. \end{aligned}$$

Berry-Esseen theorem and probability bound

The proof is completed by

$$
\begin{aligned}
\eta &= u\,\frac{2}{T\sqrt{2\pi}} \\
&\leq 3.26369\frac{2}{T\sqrt{2\pi}} \\
&= 6.52738\,\frac{(n-1)}{\sqrt{\pi}(2n-3\sqrt{2})}\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\
&= \frac{6.52738}{2\sqrt{\pi}}\left(1+\frac{3\sqrt{2}-2}{2n-3\sqrt{2}}\right)\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\
&\leq \frac{6.52738}{2\sqrt{\pi}}\left(1+\frac{3\sqrt{2}-2}{6-3\sqrt{2}}\right)\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\
&= 4.19115\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\
&\leq 5\frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}}.
\end{aligned}
$$

□