Section 37

Brownian Motion

Po-Ning Chen, Professor

Institute of Communications Engineering

National Chiao Tung University

Hsin Chu, Taiwan 30010, R.O.C.

History

- In 1827, the English botanist Robert Brown noticed that pollen grains suspended in water jiggled about under the lens of the microscope. Even more remarkable was the fact that pollen grains that had been stored for a century moved in the same way.
- In 1889, G.L. Gouy found that the "Brownian" movement was more rapid for smaller particles.
- In 1900, F.M. Exner undertook the first quantitative studies, measuring how the motion depended on temperature and particle size.
- In 1877, the first good explanation of Brownian movement was advanced by Desaulx:

  > "In my way of thinking, the phenomenon is a result of thermal molecular motion in the liquid environment (of the particles)."

  This is indeed the case. A suspended particle is constantly and randomly bombarded from all sides by molecules of the liquid. If the particle is very small, the hit it takes from one side will be stronger than the bumps from other side, causing it to jump. These small random jumps are what make up Brownian motion.

History

- In 1905, the first mathematical theory of Brownian motion was developed by Einstein. For this work, he received the Nobel prize.

Definition

Definition A *Brownian motion* or *Wiener process* is a stochastic process $\{W_t, t \geq 0\}$ (defined on some probability space $(\Omega, \mathcal{F}, P)$) with three properties:

1. **Start at 0**: $\Pr[W_0 = 0] = 1$;
2. **Independent increment**: If $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k$, then
$$\Pr[W_{t_i} - W_{t_{i-1}} \in \mathcal{H}_i, i \leq k] = \prod_{i \leq k} \Pr[W_{t_i} - W_{t_{i-1}} \in \mathcal{H}_i];$$
3. **Gaussian increment**: $W_t - W_s$ for $0 \leq s < t$ is normally distributed with mean 0 and variance $(t - s)$.

- The process is named after the nineteenth-century botanist Robert Brown.
- He is the one who first described such a random movement.
- The process is also named after Norbert Wiener who contributes the mathematical foundations of the theory of this kind of random motion.

Definition

Discussions on the properties

1. **Start at 0**: By convention.
2. **Independent increment**: The displacement is lack of memory. The particle undergos during $[t_0, t_{k-1}]$ has no influence on the displacement $W_{t_k} - W_{t_{k-1}}$.
3. **Gaussian increment**:
   - Mean 0 reflects that the particle is as likely as to go up as to go down.
   - The variance grows linearly as the length of the interval increases.

Fundamental results on moments of Brownian motion

- $E[W_t] = E[W_t - W_0] = 0.$
- $E[W_t^2] = E[(W_t - W_0)^2] = t.$
- For $0 \le s < t$,

$$\begin{aligned} E[W_s W_t] &= E[W_s(W_t - W_s) + W_s^2] \\ &= E[(W_s - W_0)(W_t - W_s) + W_s^2] \\ &= E[W_s - W_0]E[W_t - W_s] + E[W_s^2] \\ &= s \\ &= \min\{s, t\}. \end{aligned}$$

Fundamental results on dist. of Brownian motion

- The density of $(W_{t_1}, W_{t_2}, \ldots, W_{t_k})$ is:

$$f_{t_1,\ldots,t_k}(w_1,\ldots,w_k) = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left\{-\frac{(w_i - w_{i-1})^2}{2(t_i - t_{i-1})}\right\},$$

where $t_0 = w_0 = 0$ and $0 < t_1 < t_2 < \cdots < t_k$.

Proof: By the independent Gaussian distributions of $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}})$, and

$$\vec{I}_{k\times 1} = \begin{bmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ W_{t_3} - W_{t_2} \\ W_{t_4} - W_{t_3} \\ \vdots \\ W_{t_{k-1}} - W_{t_{k-2}} \\ W_{t_k} - W_{t_{k-1}} \end{bmatrix} = \mathbb{T}_{k\times k}\vec{W}_{k\times 1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} W_{t_1} \\ W_{t_2} \\ W_{t_3} \\ W_{t_4} \\ \vdots \\ W_{t_{k-1}} \\ W_{t_k} \end{bmatrix},$$

we obtain:

$$
\begin{aligned}
f_{\vec{W}}(\vec{w}) &= f_{\vec{I}}(\mathbb{T}\vec{w})|J(\vec{w})| \\
&= f_{\vec{I}}(\mathbb{T}\vec{w})\left|\begin{bmatrix} \frac{\partial T_1(\vec{w})}{\partial w_{t_1}} & \frac{\partial T_1(\vec{w})}{\partial w_{t_2}} & \cdots & \frac{\partial T_1(\vec{w})}{\partial w_{t_k}} \\ \frac{\partial T_2(\vec{w})}{\partial w_{t_1}} & \frac{\partial T_2(\vec{w})}{\partial w_{t_2}} & \cdots & \frac{\partial T_2(\vec{w})}{\partial w_{t_k}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial T_k(\vec{w})}{\partial w_{t_1}} & \frac{\partial T_k(\vec{w})}{\partial w_{t_2}} & \cdots & \frac{\partial T_k(\vec{w})}{\partial w_{t_k}} \end{bmatrix}\right| \\
&= f_{\vec{I}}(\mathbb{T}\vec{w})\left|\begin{bmatrix} \frac{\partial w_{t_1}}{\partial w_{t_1}} & \frac{\partial w_{t_1}}{\partial w_{t_2}} & \cdots & \frac{\partial w_{t_1}}{\partial w_{t_k}} \\ \frac{\partial(w_{t_2}-w_{t_1})}{\partial w_{t_1}} & \frac{\partial(w_{t_2}-w_{t_1})}{\partial w_{t_2}} & \cdots & \frac{\partial(w_{t_2}-w_{t_1})}{\partial w_{t_k}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial(w_{t_k}-w_{t_{k-1}})}{\partial w_{t_1}} & \frac{\partial(w_{t_k}-w_{t_{k-1}})}{\partial w_{t_2}} & \cdots & \frac{\partial(w_{t_k}-w_{t_{k-1}})}{\partial w_{t_k}} \end{bmatrix}\right| \\
&= f_{\vec{I}}(\mathbb{T}\vec{w})\,|\mathbb{T}| \\
&= f_{\vec{I}}(\mathbb{T}\vec{w}).
\end{aligned}
$$

■

Brownian motion and law of large numbers

- If $X_1, X_2, \ldots, X_n$ are independent Gaussian distributed with mean 0 and variance $t_1, t_2 - t_1, \ldots, t_k - t_{k-1}$, and $S_k = X_1 + X_2 + \cdots + X_k$, then $(S_1, S_2, \ldots, S_k)$ has the same distribution as $(W_{t_1}, W_{t_2}, \ldots, W_{t_k})$.

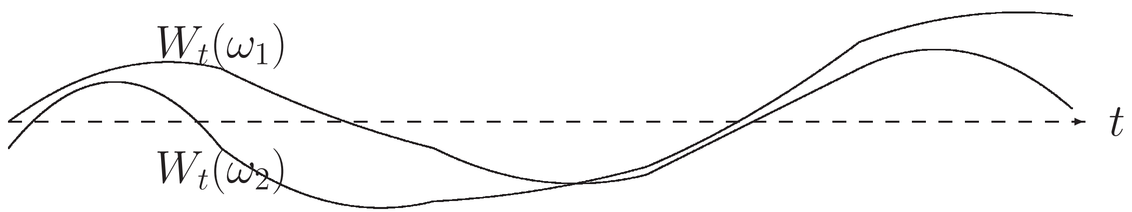
Path function of Brownian motion

- Sometimes, the Brownian motion will add another required property that relies on the inherited probability space $(\Omega, \mathcal{F}, P)$.

  4. For each $\omega \in \Omega$, $W_t(\omega)$ is continuous in t and $W_0(\omega) = 0$.

Notably, X is a random variable defined on $(\Omega, \mathcal{F}, P)$; but, $X(\omega)$, as a function mapping, is a deterministic value for each $\omega \in \Omega$, not random at all !

  - $W_t(\omega)$ is a deterministic function in t.
  - It is a path trace of an article; so, the property above is usually referred as *continuity of paths*.



Measurable processes

Theorem 37.2 Brownian motion is measurable.

- $\{W_t, t \in \mathcal{T}\}$ is a process defined over $(\Omega, \mathcal{F}, P)$, where $\mathcal{T}$ is a Borel subset of real line.
- So $W_t(\omega)$ is a **deterministic** function mapping from $\mathcal{T} \times \Omega$ to $\Re$.

- Let $\mathcal{B}(\mathcal{T})$ be the σ-field of Borel set $\mathcal{T}$.
- Then the above theorem says that $W_t(\omega)$ is $(\mathcal{B}(\mathcal{T}) \times \mathcal{F})/\mathcal{B}$-measurable.

Irregularity of Brownian motion paths

Define $\tilde{W}_t(\omega) = \dfrac{1}{c} W_{c^2 t}(\omega)$ for some positive constant c.

Then it can be shown that $\{\tilde{W}_t, t \geq 0\}$ is also a Brownian motion, and has the same distribution as $\{W_t, t \geq 0\}$.

1. **Start at 0**: $\Pr[\tilde{W}_0 = 0] = \Pr\left[\dfrac{1}{c} W_0 = 0\right] = 1$;
2. **Independent increment**: If $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k$, then

$$
\begin{aligned}
\Pr[\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}} \in \mathcal{H}_i, i \leq k] &= \Pr\left[\frac{1}{c}\left(W_{c^2 t_i} - W_{c^2 t_{i-1}}\right) \in \mathcal{H}_i, i \leq k\right] \\
&= \Pr\left[W_{c^2 t_i} - W_{c^2 t_{i-1}} \in c\mathcal{H}_i, i \leq k\right] \\
&= \prod_{i \leq k} \Pr[W_{c^2 t_i} - W_{c^2 t_{i-1}} \in c\mathcal{H}_i] \\
&= \prod_{i \leq k} \Pr\left[\frac{1}{c}\left(W_{c^2 t_i} - W_{c^2 t_{i-1}}\right) \in \mathcal{H}_i\right] \\
&= \prod_{i \leq k} \Pr\left[\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}} \in \mathcal{H}_i\right]
\end{aligned}
$$

3. **Gaussian increment**:

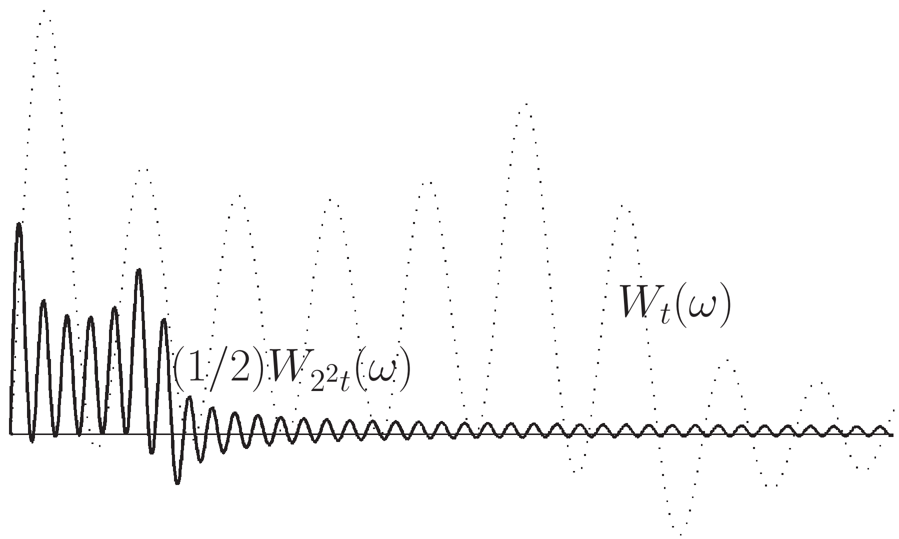
$$\tilde{W}_t - \tilde{W}_s \;=\; \frac{1}{c}\left(W_{c^2t} - W_{c^2s}\right)$$

for $0 \le s < t$ is normally distributed with mean 0 and variance $\frac{1}{c^2}(c^2t - c^2s) = t - s$.

4. **Continuity**: For each $\omega \in \Omega$, $\tilde{W}_t(\omega) = \frac{1}{c}W_{c^2t}(\omega)$ is continuous in t and $\tilde{W}_0(\omega) = \frac{1}{c}W_0(\omega) = 0$.

Irregularity of Brownian motion paths

- The “distribution” (actually, only the three statistical properties) of $\{W_t, t \geq 0\}$ remains the same after contracting the time scale by the factor c^2, but only contracting the space scale by a factor of c.



- This shows that $W_t(\omega)$, although continuous, must be *highly irregular*.

Irregularity of Brownian motion paths

What does **irregularity** mean?

- Suppose that the path $W_t(\omega)$ has slope exceeding 1 at some interval $[0, c]$.
- Then $\tilde{W}_t(\omega)$ has slope exceeding c at interval $[0, 1/c]$.

$$\begin{aligned}\frac{\tilde{W}_{t+h}(\omega) - \tilde{W}_t(\omega)}{h} &= \frac{\frac{1}{c}\left(W_{c^2t+c^2h}(\omega) - W_{c^2t}(\omega)\right)}{h} \\ &= c \times \frac{W_{t'+h'}(\omega) - W_{t'}(\omega)}{h'}, \text{ where } h' = c^2h \text{ and } t' = c^2t.\end{aligned}$$

- $t' \in [0, c] \Leftrightarrow t \in \dfrac{1}{c^2}[0, c] = [0, 1/c]$.
- What happen to $[f(x+h) - f(x)]/h = c \cdot [g(x+h) - g(x)]/h$ if $f = g$? What happen if $f \neq g$?

- Since $\tilde{W}_t$ is as disturbed as W_t in statistics, the above statement indicates that W_t must, with great probability, have arbitrarily great slopes $1/\delta$ within arbitrarily small interval $[0, \delta]$.
- So W_t is not differentiable at $t = 0$.
- Indeed, W_t is with probability 1 nowhere differentiable.

Irregularity of Brownian motion paths

What does **irregularity** mean?

- Nowhere differentiability with probability 1 of W_t can also be anticipated by the **independent increment**, where an abrupt change may occur at the next time instant.

Irregularity of Brownian motion paths

An interesting case occurs when letting $c = 1/t$, namely,

$$\hat{W}_t(\omega) = \begin{cases} \left(\frac{1}{c} W_{c^2 t}(\omega) = \right) & tW_{1/t}(\omega), \quad \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

This exchanges those function values between $[1, \infty)$ and $(0, 1]$.

As a consequence, $\{\hat{W}_t, t \geq 0\}$ has the same "distribution" (only three statistical properties) as $\{W_t, t \geq 0\}$.

1. **Start at 0**: $\Pr[\hat{W}_0 = 0] = 1$ by definition;

Irregularity of Brownian motion paths

2. **Independent increment**: If $0 < t_1 \le t_2 \le t_3$, then

$$
\begin{aligned}
\begin{bmatrix} \hat{W}_{t_2} - \hat{W}_{t_1} \\ \hat{W}_{t_3} - \hat{W}_{t_2} \end{bmatrix}
&= \begin{bmatrix} t_2 W_{1/t_2} - t_1 W_{1/t_1} \\ t_3 W_{1/t_3} - t_2 W_{1/t_2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & t_2 & -t_1 \\ t_3 & -t_2 & 0 \end{bmatrix} \begin{bmatrix} W_{1/t_3} \\ W_{1/t_2} \\ W_{1/t_1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & t_2 & -t_1 \\ t_3 & -t_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} W_{1/t_3} \\ W_{1/t_2} - W_{1/t_3} \\ W_{1/t_1} - W_{1/t_2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & t_2 & -t_1 \\ t_3 & -t_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} W_{1/t_3} \\ W_{1/t_2} - W_{1/t_3} \\ W_{1/t_1} - W_{1/t_2} \end{bmatrix} \\
&= \begin{bmatrix} t_2 - t_1 & t_2 - t_1 & -t_1 \\ t_3 - t_2 & -t_2 & 0 \end{bmatrix} \begin{bmatrix} W_{1/t_3} \\ W_{1/t_2} - W_{1/t_3} \\ W_{1/t_1} - W_{1/t_2} \end{bmatrix}.
\end{aligned}
$$

Irregularity of Brownian motion paths

Since $(\hat{W}_{t_2} - \hat{W}_{t_1}, \hat{W}_{t_3} - \hat{W}_{t_2})$ are zero-mean Gaussian distributed, and

$$
\begin{aligned}
E\left[(\hat{W}_{t_2} - \hat{W}_{t_1})(\hat{W}_{t_3} - \hat{W}_{t_2})\right] &= (t_3 - t_2)(t_2 - t_1)E[W^2_{1/t_3}] - t_2(t_2 - t_1)E[(W_{1/t_2} - W_{1/t_3})^2] \\
&= (t_3 - t_2)(t_2 - t_1)\frac{1}{t_3} - t_2(t_2 - t_1)\left(\frac{1}{t_2} - \frac{1}{t_3}\right) \\
&= 0,
\end{aligned}
$$

$\hat{W}_{t_2} - \hat{W}_{t_1}$ and $\hat{W}_{t_3} - \hat{W}_{t_2}$ are independent.

3. **Gaussian increment**:

$$
\hat{W}_t - \hat{W}_s \;=\; tW_{1/t} - sW_{1/s} = s(W_{1/t} - W_{1/s}) + (t-s)(W_{1/t} - W_0)
$$

for $0 \le s < t$ is normally distributed with mean 0 and variance

$$
s^2\left(\frac{1}{s} - \frac{1}{t}\right) + (t-s)^2\left(\frac{1}{t}\right) = t - s.
$$

4. **Continuity**: For each $\omega \in \Omega$, $\hat{W}_t(\omega) = tW_{1/t}(\omega)$ is continuous in t and $\hat{W}_0(\omega) = 0$.

Limit of Brownian motion

- The behavior of $W_t(\omega)$ near 0 can be studied through the behavior of $\hat{W}_t(\omega)$ near ∞.

- Since

$$\frac{\hat{W}_h(\omega) - \hat{W}_0(\omega)}{h} = W_{1/h}(\omega),$$

$\hat{W}_t(\omega)$ cannot have a derivative at $t = 0$, if $W_t(\omega)$ has no limit at $t \to \infty$.

Surprisingly, with probability 1,

$$\begin{aligned}
\liminf_{h \to 0} \frac{\hat{W}_h - \hat{W}_0}{h} &= \liminf_{n \to \infty} \frac{\hat{W}_{1/n} - \hat{W}_0}{1/n} \\
&= \liminf_{n \to \infty} W_n \\
&= \lim_{n \to \infty} \inf_{m \geq n} W_m \\
&= -\infty
\end{aligned}$$

since $\inf_{n \geq m} W_n = -\infty$ with probability one for any fixed m.

$$\liminf_{n \to \infty} a_n = \lim_{n \to \infty} \inf_{m \geq n} a_m = \sup_{n \geq 1} \inf_{m \geq n} a_n.$$

Limit of Brownian motion

Similarly,

$$\limsup_{h \to 0} \frac{\hat{W}_h - \hat{W}_0}{h} = \infty \quad \Big(\text{since } \sup_{n \geq 1} W_n = \infty\Big)$$

with probability 1.

- We now prove $\inf_{n\geq 1} W_n = -\infty$ and $\sup_{n\geq 1} W_n = \infty$ with probability one in the following slides.

Limit of Brownian motion

- **Lemma** $\Pr[\sup_{n\geq 1}|W_n| = \infty] = 1$.

Proof: For any $x > 0$,

$$\begin{aligned}
\Pr\left[\sup_{n\geq 1}|W_n| < x\right] &\leq \Pr[|W_n| < x] \\
&= \Pr\left[|W_1| < \frac{x}{\sqrt{n}}\right] \\
&= 1 - 2\Phi\left(-\frac{x}{\sqrt{n}}\right),
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cdf.
Since the upper bound holds for any n,

$$\Pr\left[\sup_{n\geq 1}|W_n| < x\right] = 0 \text{ for any } x > 0.$$

Therefore, $\sup_{n\geq 1}|W_n| = \infty$ with probability 1. $\square$

Limit of Brownian motion

- **Lemma** $\Pr[\inf_{n\geq 1} W_n = -\infty] = 1$ and $\Pr[\sup_{n\geq 1} W_n = \infty] = 1$.

  Proof:

  – $W_n = X_1 + \cdots + X_n$, where $\{X_k = W_k - W_{k-1}\}$ are i.i.d. Gaussian.

  – Observe that

$$\mathcal{A} = \left\{\omega \in \Omega : \sup_{n\geq 1} W_n(\omega) < \infty\right\} \;=\; \left\{\omega \in \Omega : \sup_{n\geq 1}\left(W_{n+k}(\omega) - W_k(\omega)\right) < \infty\right\}$$

Also observe that

$$\begin{aligned}\mathcal{A} = \left\{\omega \in \Omega : \sup_{n\geq 1} W_n(\omega) < \infty\right\} \;&=\; \bigcap_{j=1}^{\infty}\left\{\omega \in \Omega : \max_{1\leq n\leq j} W_n(\omega) < \infty\right\} \\ &\subset\; \bigcap_{j=1}^{k}\left\{\omega \in \Omega : \max_{1\leq n\leq j} W_n(\omega) < \infty\right\} = \mathcal{A}_k.\end{aligned}$$

Then, by the fact that $\mathcal{A}$ (determined by $W_{1+k} - W_k$, $W_{2+k} - W_k$, $\cdots$, or equivalently, $X_{k+1}, X_{k+2}, \cdots$) is independent of $\mathcal{A}_k$ (determined by $W_1, W_2, \cdots, W_k$, or equivalent, $X_1, X_2, \cdots, X_k$), we obtain:

$$P(\mathcal{A}) = P(\mathcal{A} \cap \mathcal{A}) \;\leq\; P(\mathcal{A} \cap \mathcal{A}_k) = P(\mathcal{A})P(\mathcal{A}_k).$$

Now if $P(\mathcal{A}) > 0$, then $P(\mathcal{A}_k) = 1$ for every k; hence,

$$P\left(\bigcap_{k=1}^{\infty} \mathcal{A}_k\right) = P(\mathcal{A}) = 1.$$

This concludes that either $P(\mathcal{A}) = 1$ or $P(\mathcal{A}) = 0$.

– Since $-X_n$ has the same distribution as X_n,

$$\Pr\left[\sup_{n\geq 1} W_n < \infty\right] = \Pr\left[\sup_{n\geq 1}(-W_n) < \infty\right] = \Pr\left[\inf_{n\geq 1} W_n > -\infty\right].$$

– Suppose $\Pr\left[\sup_{n\geq 1} W_n < \infty\right] = \Pr\left[\inf_{n\geq 1} W_n > -\infty\right] = 1$.
Then

$$\begin{aligned} 1 &= \Pr\left[\sup_{n\geq 1} W_n < \infty \text{ and } \inf_{n\geq 1} W_n > -\infty\right] \\ &= \Pr\left[\sup_{n\geq 1} |W_n| < \infty\right] = 0, \text{ by the previous lemma.} \end{aligned}$$

A desired contradiction is therefore obtained. □

- Consequently, W_t is with probability 1 not differentiable at $t = 0$.

Limit of Brownian motion

- In fact, since W_t can be viewed as starting at $t = t_0$ with a new origin locating at W_{t_0}, W_t is with probability 1 not differentiable at any t. (This is what **Theorem 37.3** states.)

Limit of Brownian motion

- A nowhere-differentiable path or trace of an article indicates no "*velocity*" at any time.
- This particle can possibly travel an *infinite distance* at finite time.
- The Brownian motion model, like white noise, thus *does not in its fine structure represent physical reality*.
- By the way, another interesting outcome of Brownian motion is that $W_t = 0$ infinitely often in t, namely W_t changes signs infinitely often.

  In fact, $W_t = a$ infinitely often in t for any $a \in \Re$.

Another transformation of Brownian motion

- For any $t_0 \geq 0$ fixed,

$$W_t^* = W_{t+t_0} - W_{t_0}$$

is a Brownian motion.

- Let $\boldsymbol{\tau}$ be a non-negative random variable.
Then

$$W_t^{**} = W_{t+\boldsymbol{\tau}} - W_{\boldsymbol{\tau}}$$

is a Brownian motion.

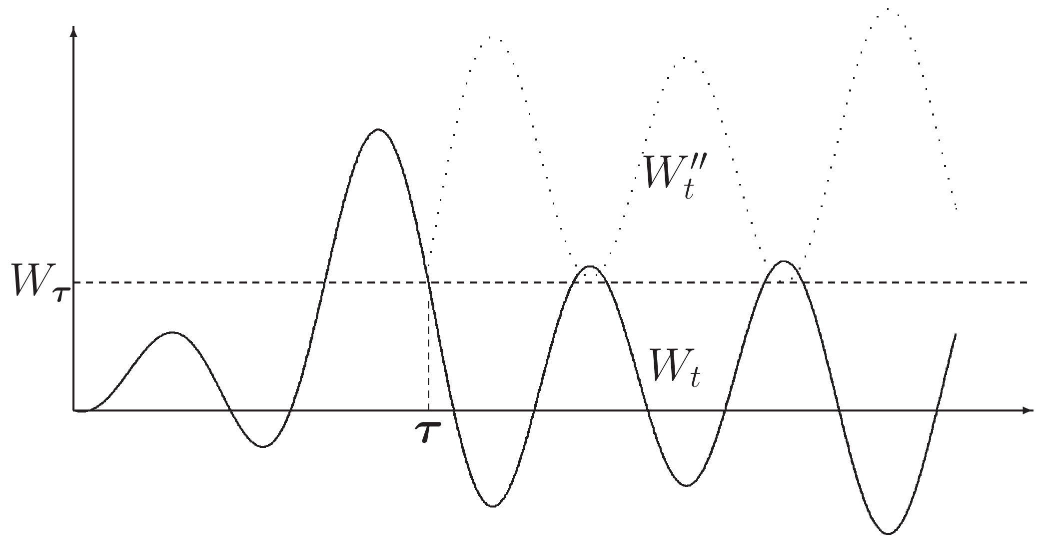
$\boldsymbol{\tau}$ is often named as the *stopping time*.

This is what stated in **Theorem 37.5 (Strong Markov property)**.

The reflection principle

For a stopping time $\boldsymbol{\tau}$ (non-negative random variable), define

$$W_t'' = \begin{cases} W_t, & \text{if } t \le \boldsymbol{\tau}; \\ W_{\boldsymbol{\tau}} - (W_t - W_{\boldsymbol{\tau}}), & \text{if } t \ge \boldsymbol{\tau}. \end{cases}$$



As anticipated, W_t'' is a Brownian motion.

Skorohod embedding theorems

Theorem 37.6 (Skorohod embedding theorem) Suppose that X is a random variable with mean 0 and finite variance. Then there is a stopping time $\boldsymbol{\tau}$ (random variable) such that $W_{\boldsymbol{\tau}}$ has the same distribution as X, and

$$E[\boldsymbol{\tau}] = E[X^2] \text{ and } E[\boldsymbol{\tau}^2] \leq 4E[X^4].$$

Theorem 37.7 (Skorohod embedding theorem) Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and finite variance.
Let $S_n = X_1 + \cdots + X_n$.
Then there is a non-decreasing sequence of stopping times $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots$ such that

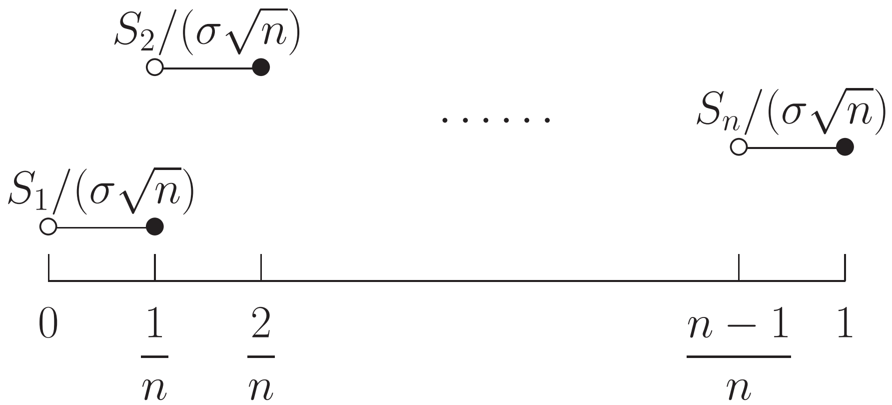
1. $W_{\boldsymbol{\tau}_n}$ has the same distribution as S_n, and
2. $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1, \boldsymbol{\tau}_3 - \boldsymbol{\tau}_2, \ldots$ are i.i.d. with

$$E[\boldsymbol{\tau}_n - \boldsymbol{\tau}_{n-1}] = E[X_1^2]$$

and

$$E[(\boldsymbol{\tau}_n - \boldsymbol{\tau}_{n-1})^2] \leq 4E[X_1^4].$$

Functional central limit theorem



Theorem 37.8 (**functional central limit theorem**) Define for each integer n a random process $\{Y_t(n), 0 \le t \le 1\}$ as:

$$Y_t(n) = \frac{S_k}{\sigma\sqrt{n}} \quad \text{for } \frac{k-1}{n} < t \le \frac{k}{n}, \text{ and } k = 1, 2, \ldots, n,$$

where $S_n = X_1 + X_2 + \cdots + X_n$ for i.i.d. $X_1, X_2, \cdots$. If $E[X_1^4] < \infty$, there exist $\{Z_t(n), 0 \le t \le 1\}$ and $\{W_t(n), 0 \le t \le 1\}$ such that

1. $\{Z_t(n), 0 \le t \le 1\}$ and $\{Y_t(n), 0 \le t \le 1\}$ have the same finite dimensional distribution;
2. $\{W_t(n), 0 \le t \le 1\}$ is a Brownian motion;
3. $\lim_{n\to\infty} \Pr\left[\sup_{0\le t\le 1} |Z_t(n) - W_t(n)| \ge \varepsilon\right] = 0$.