Ninness's Strong Law of Large Numbers

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The law of large numbers

- The employment of “Strong Low of Large Numbers (SLLN)” is instrumental to the analysis of system estimation and identification strategies.
- However, the condition (for its validity) such as independence or uncorrelatedness of random components is quite restrictive from an engineering standpoint.
- In his paper, Brett Ninness shows that the SLLN is valid even for possibly non-stationary random components under very general dependence structure.
  - B. Ninness, “Strong laws of large numbers under weak assumptions with applications,” *IEEE Trans. Automatic Control*, vol. 45, no. 11, pp. 2117–2122, 2000.

Ninness Strong Law of Large Numbers

Theorem Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables, not necessarily zero mean, and with arbitrary correlation structure (not necessarily stationary) that is characterized by the existence of $C < \infty$ and $\beta > 1$ such that

$$\sum_{k=i+1}^{j} \sum_{\ell=i+1}^{j} E[X_k X_\ell] \leq C(j-i)^{\beta} \text{ for every } 0 \leq i \leq j.$$

Then for any $\alpha > \beta/2$,

$$\frac{1}{n^{\alpha}} \sum_{k=1}^{n} X_k \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty.$$

Proof:

- Define $S_n = \sum_{k=1}^{n} X_n$, and for any n, choose an integer $m = \lfloor \log_2(n) \rfloor$ such that

  $$2^m \leq n < 2^{m+1}.$$

  Then

  $$\frac{1}{n^{\alpha}} |S_n| \leq \frac{1}{2^{m\alpha}} \max_{0 \leq k \leq 2^{m+1}} |S_k|.$$

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Refined version of **Theorem 12.2** in [P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1968]. For arbitrary random variables $\{X_n\}_{n=1}^{\infty}$, if for some $\beta > 1$, there exists a set of non-negative numbers $\{u_n\}_{n=1}^{\infty}$ such that

$$E\left[|S_j - S_i|^2\right] \le \left(\sum_{k=i}^{j} u_k\right)^{\beta} \quad \text{for every } 0 \le i \le j \le N,$$

where $S_j = \sum_{k=1}^{j} X_k$ and $S_0 = 0$, then there exists a $K < \infty$ that is independent of N such that for any $\lambda > 0$,

$$\Pr\left\{\max_{0\le \ell \le N} |S_\ell| \ge \lambda\right\} \le \frac{K}{\lambda^2}\left(\sum_{k=1}^{N} u_k\right)^{\beta}.$$

Furthermore, the constant K can be taken as $K = 4\left(1 + \frac{2^{(\beta-1)}}{\left(2^{(\beta-1)/3}-1\right)^3}\right)$.

- Taking $u_n = C^{1/\beta}$ for every n in the above refined theorem yields:

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If for some $\beta > 1$,

$$E\left[|S_j - S_i|^2\right] \le C\,(j-i)^\beta \text{ for every } 0 \le i \le j,$$

then there exists a $K < \infty$ that is only dependent on β such that for any $\lambda > 0$ and any N,

$$\Pr\left\{\max_{0\le \ell \le N} |S_\ell| \ge \lambda\right\} \le \frac{KC}{\lambda^2} N^\beta.$$

Hence for $\lambda = \varepsilon 2^{m\alpha}$ and $N = 2^{m+1}$,

$$\Pr\left\{\max_{0\le k \le 2^{m+1}} |S_k| \ge \varepsilon 2^{m\alpha}\right\} \le \frac{KC}{\varepsilon^2 2^{2m\alpha}} (2^{m+1})^\beta = \frac{KC2^\beta}{\varepsilon^2 2^{(2\alpha-\beta)m}}$$

since

$$E[|S_j - S_i|^2] = \sum_{k=i+1}^{j} \sum_{\ell=i=1}^{j} E[X_k X_\ell] \le C\,(j-i)^\beta$$

for every $0 \le i \le j$.

- As a result, for $\alpha > \beta/2$,

$$\sum_{m=1}^{\infty} \Pr\left\{ \max_{0\le k\le 2^{m+1}} |S_k| \ge \varepsilon 2^{m\alpha} \right\} \le \frac{KC2^{\beta}}{\varepsilon^2} \sum_{m=1}^{\infty} 2^{-(2\alpha-\beta)m} = \frac{KC2^{\beta}}{\varepsilon^2(2^{2\alpha-\beta}-1)} < \infty.$$

By the first Borel-Cantelli lemma, we obtain that with probability 1

$$\frac{1}{2^{m\alpha}} \max_{0\le k\le 2^{m+1}} |S_k| \ge \varepsilon$$

is valid only for finitely many m.

Theorem 4.3 (The First Borel-Cantelli Lemma) If $\sum_{n=1}^{\infty} P(A_n)$ converges, then

$$P\left(\limsup_{n\to\infty} A_n\right) = P(A_n \text{ i.o.}) = 0.$$

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- Since ε can be made arbitrarily small,

$$\limsup_{m\to\infty} \frac{1}{2^{m\alpha}} \max_{0\le k\le 2^{m+1}} |S_k| = 0 \text{ with probability 1.}$$

The theorem holds by noting that

$$\limsup_{n\to\infty} \frac{1}{n^\alpha}|S_n| \le \limsup_{n\to\infty} \frac{1}{2^{\lfloor \log_2(n)\rfloor\alpha}} \max_{0\le k\le 2^{\lfloor \log_2(n)\rfloor+1}} |S_k| = \limsup_{m\to\infty} \frac{1}{2^{m\alpha}} \max_{0\le k\le 2^{m+1}} |S_k|.$$

$\square$

Example 1

- Let

$$y_n = \theta + v_n,$$

where θ is an unknown constant to be estimated based upon the observations of $\{y_n\}$, and

$$E[v_n v_m] = \frac{1 + |n-m|^p}{1 + |n-m|^q} \text{ for some } q > 1 \text{ and } 0 \le p < 1.$$

- Use the estimator $\hat{\theta} = \frac{1}{n}\sum_{k=1}^{n} y_k$.
- Then the estimation error $\theta_e = \hat{\theta} - \theta = \frac{1}{n}\sum_{k=1}^{n} v_k$.
- Verify by defining $S_j = v_1 + v_2 + \cdots + v_j$ that for $j > i$,

$$\begin{aligned} E[|S_j - S_i|^2] &= \sum_{k=i+1}^{j} \sum_{\ell=i+1}^{j} E[v_k v_\ell] \\ &= \sum_{k=i+1}^{j} \sum_{\ell=i+1}^{j} \frac{1 + |k-\ell|^p}{1 + |k-\ell|^q} \\ &= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{1 + |k-\ell|^p}{1 + |k-\ell|^q} \quad (m = j - i) \end{aligned}$$

Example 1

$$
\begin{aligned}
&= m + 2\sum_{k=1}^{m}\sum_{\ell=1}^{k-1}\frac{1+(k-\ell)^p}{1+(k-\ell)^q} \\
&\le m + 2\sum_{k=1}^{m}\sum_{\ell=1}^{k-1}\frac{2(k-\ell)^p}{1+(k-\ell)^q} \quad (\text{since } p \ge 0) \\
&\le m + 4\sum_{k=1}^{m}\sum_{\ell=1}^{k-1}\frac{m^p}{1+(k-\ell)^q} \quad (\text{since } \max_{1\le k\le m}\max_{1\le \ell\le k-1}(k-\ell) = (m-1) \le m) \\
&= m + 4m^p\sum_{k=1}^{m}\sum_{\ell=1}^{k-1}\frac{1}{1+(k-\ell)^q} \\
&= m + 4m^p\sum_{u=1}^{m}(m-u)\cdot\frac{1}{1+u^q} \\
&\le m + 4m^{p+1}\sum_{u=1}^{m}\frac{1}{1+u^q} \quad (\text{because } m-u \le m \text{ for } 1 \le u \le m) \\
&\le m + 4m^{p+1}\int_0^m \frac{1}{1+u^q}du \quad (\text{because } \frac{1}{1+u^q} \text{ nonincreasing for } u \ge 0)
\end{aligned}
$$

Example 1

$$
\begin{aligned}
&\le \; m + 4m^{p+1} \int_0^m \frac{1}{1+u^q} du \\
&= \; m + 4m^{p+1} \left(\int_0^1 \frac{1}{1+u^q} du + \int_1^m \frac{1}{1+u^q} du \right) \\
&\le \; m + 4m^{p+1} \left(1 + \int_1^m \frac{1}{1+u^q} du \right) \quad \left(\text{since } \int_0^1 \frac{1}{1+u^q} du \le 1\right) \\
&\le \; m + 4m^{p+1} \left(1 + \int_1^m \frac{1}{u^q} du \right) \\
&= \; m + 4m^{p+1} \left(\frac{m^{1-q} - q}{1-q} \right) \\
&= \; m + \frac{4}{1-q} m^{p-q+2} - \frac{4q}{1-q} m^{p+1} \\
&\le \; m^{p+1} + 0 + \frac{4q}{q-1} m^{p+1} \quad (\text{because } q > 1 \text{ and } p \ge 0) \\
&= \; \frac{5q-1}{q-1} m^{p+1}
\end{aligned}
$$

Example 1

Hence, as $(\alpha =) 1 > (p+1)/2 (= \beta/2)$,

$$\theta_e = \hat{\theta} - \theta = \frac{1}{n^\alpha} \sum_{k=1}^{n} v_k = \frac{1}{n} \sum_{k=1}^{n} v_k \rightarrow 0 \text{ with probability } 1.$$

Example 2

- Suppose $\{v_n\}$ (in Example 1) is wide-sense stationary with bounded power spectral density. Then, by denoting the bound for the power spectral density by P,

$$\begin{aligned}
E[|S_j - S_i|^2] &= \sum_{k=i+1}^{j}\sum_{\ell=i+1}^{j} E[v_k v_\ell] \\
&= \sum_{k=1}^{m}\sum_{\ell=1}^{m} E[v_k v_\ell] \quad (m = j - i) = \sum_{k=1}^{m}\sum_{\ell=1}^{m} R_v(k-\ell) \\
&\le \sum_{k=1}^{m}\sum_{\ell=1}^{m} \int_{-1/2}^{1/2} P e^{i2\pi(k-\ell)f} df \\
&= P \sum_{k=1}^{m}\sum_{\ell=1}^{m} \int_{-1/2}^{1/2} e^{i2\pi(k-\ell)f} df \\
&= Pm.
\end{aligned}$$

- Hence, for $\alpha = 1 > 1/2 = \beta/2$,

$$\theta_e = \hat{\theta} - \theta = \frac{1}{n}\sum_{k=1}^{n} v_k \to 0 \text{ with probability 1.}$$

Example 2

$$\begin{cases} S_v(\omega) = \displaystyle\sum_{k=-\infty}^{\infty} R_v[k] e^{-\imath \omega k} \\ R_v[k] = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} S_v(\omega) e^{\imath \omega k} d\omega \end{cases} \equiv \begin{cases} S_v(f) = \displaystyle\sum_{k=-\infty}^{\infty} R_v[k] e^{-\imath 2\pi k f} \\ R_v[k] = \displaystyle\int_{-1/2}^{1/2} S_v(f) e^{\imath 2\pi k f} df \end{cases}$$