Section 28

Infinitely Divisible Distributions

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Poisson and Normal as two limit laws

Theorem 23.2 $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,r_n}$ are independent random variables.
$\Pr[Z_{n,k} = 1] = p_{n,k}$ and $\Pr[Z_{n,k} = 0] = 1 - p_{n,k}$.
Then

$$\left.\begin{array}{l} (i)\ \lim_{n\to\infty} \sum_{k=1}^{r_n} p_{n,k} = \lambda \\ (ii)\ \lim_{n\to\infty} \max_{1\le k\le r_n} p_{n,k} = 0 \end{array}\right\} \Rightarrow \Pr\left[\sum_{k=1}^{r_n} Z_{n,k} = i\right] \to e^{-\lambda}\frac{\lambda^i}{i!} \text{ for } i = 0, 1, 2, \ldots.$$

Theorem 27.2 For an array of independent zero-mean random variables $X_{n,1}, \ldots, X_{n,r_n}$, if Lindeberg's condition holds for all positive ε, then

$$\frac{S_n}{s_n} \Rightarrow N,$$

where $S_n = X_{n,1} + \cdots + X_{n,r_n}$.

Limit laws

- We have learned thus far that Poisson and Normal are two limit laws for sum of independent array variables.
- **Question:** What are the class of all possible limit laws for sum of independent triangular array variables?

Infinitely divisible distribution

Definition (Infinitely divisible) A distribution F is *infinitely divisible* if for each n, there exists a distribution function F_n such that F is the n-fold convolution $\underbrace{F_n * \cdots * F_n}_{n \text{ copies}}$ of F_n.

- **Question:** What are the class of all possible limit laws for sum of independent triangular array variables?
- **Answer:** The class of possible limit laws consists of the infinitely divisible distributions.

Infinitely divisible distribution

Theorem 28.1 For any finite (non-negative) measure (not necessarily probability measure),

$$\varphi(t) = \exp\left\{\int_{\Re}\left(e^{itx} - 1 - itx\right)\frac{1}{x^2}\mu(dx)\right\}$$

is the characteristic function of an infinitely divisible distribution with mean 0 and variance $\mu(\Re)$.

- μ is named the *canonical measure.*
- $\exp\left\{\int_{\Re}\left(e^{itx} - 1 - itx\right)\frac{1}{x^2}\mu(dx)\right\}$ is named the *canonical representation* of infinitely divisible distributions (of zero mean and finite variance).

Theorem 28.2 Every infinitely divisible distribution with mean 0 and finite variance is the limit law of $S_n = X_{n,1} + \cdots + X_{n,r_n}$ for some independent triangular array satisfying:

1. $E[X_{n,k}] = 0$;
2. $\lim_{n\to\infty} \max_{1\le k\le r_n} E[X_{n,k}^2] = 0$;
3. $\sup_{n\ge 1} s_n^2 < \infty$, where $s_n^2 = \sum_{k=1}^{r_n} E[X_{n,k}^2]$.

Theorem 27.2 For an array of independent zero-mean random variables $X_{n,1}, \ldots, X_{n,r_n}$, if Lindeberg's condition holds for all positive ε, then

$$\frac{S_n}{s_n} \Rightarrow N,$$

where $S_n = X_{n,1} + \cdots + X_{n,r_n}$.

- The case considered in Theorem 27.2 is a special case among those considered in Theorem 28.2.
- Specifically, let $\tilde{X}_{n,k} = X_{n,k}/s_n$, where $X_{n,k}$ and s_n are defined in Theorem 27.2. Then the conditions considered in Theorem 28.2 becomes the limit of $\tilde{S}_n = \tilde{X}_{n,k} + \cdots + \tilde{X}_{n,r_n} = S_n/s_n$, and
  1. $E[\tilde{X}_{n,k}] = E[X_{n,k}/s_n] = 0$;
  2. $\lim_{n\to\infty} \max_{1\le k\le r_n} E[\tilde{X}^2_{n,k}] = \lim_{n\to\infty} \max_{1\le k\le r_n} \dfrac{E[X^2_{n,k}]}{s_n^2} = 0$;
  3. $\sup_{n\ge 1} \tilde{s}_n^2 < \infty$, where $\tilde{s}_n^2 = \sum_{k=1}^{r_n} E[\tilde{X}^2_{n,k}] = \sum_{k=1}^{r_n} \dfrac{E[X^2_{n,k}]}{s_n^2} = 1$.

Example 28.1 μ is a point mass at the origin, and $\mu\{0\} = \sigma^2$.

$$\begin{aligned}
\varphi(t) &= \exp\left\{\int_{\Re}\left(e^{itx} - 1 - itx\right)\frac{1}{x^2}\mu(dx)\right\} \\
&= \exp\left\{\sigma^2 \lim_{x\to 0}\frac{\left(e^{itx} - 1 - itx\right)}{x^2}\right\} \\
&= \exp\left\{\sigma^2 \lim_{x\to 0}\frac{\left(ite^{itx} - it\right)}{2x}\right\} \\
&= \exp\left\{\sigma^2 \lim_{x\to 0}\frac{\left((it)^2 e^{itx}\right)}{2}\right\} \\
&= \exp\left\{-\frac{\sigma^2 t^2}{2}\right\}.
\end{aligned}$$

Hence, a central normal distribution with variance $\sigma^2 = \mu(\Re)$ is infinitely divisible.

Example 28.2 μ consists of a point mass λx^2 at some $x \neq 0$.

$$\begin{aligned}\varphi(t) &= \exp\left\{\int_{\Re}\left(e^{itx}-1-itx\right)\frac{1}{x^2}\mu(dx)\right\}\\ &= \exp\left\{\lambda\left(e^{itx}-1-itx\right)\right\},\end{aligned}$$

which is the characteristic function of $x(Z_\lambda - \lambda)$, where Z_λ has Poisson distribution with mean λ.

Notably, the variance (2nd moment) of $x(Z_\lambda - \lambda)$ is equal to $\lambda x^2 = \mu(\Re)$.

For any n, its cdf F can be represented by the n-fold convolution of F_n for which F_n is the cdf of $x(Z_{\lambda/n} - \lambda/n)$.

Proof of Theorem 28.1

(Proof of $\varphi(t)$ is a characteristic function)

- For any **finite** measure μ, define a new measure μ_k which has point mass $\mu(j2^{-k}, (j+1)2^{-k}]$ at $j2^{-k}$ for $j = 0, \pm 1, \pm 2, \ldots, \pm 2^{2k}$.

  Then μ_k converges to μ vaguely.

  Here, **finite** is a key because this property may not be true for **infinite** measure.

- **Lemma** Suppose that $\mu_n \xrightarrow{v} \mu$ and $\sup_{n\geq 1} \mu_n(\Re) < \infty$. Then $\lim_{n\to\infty} \int_\Re f(x)\mu_n(dx) = \int_\Re f(x)\mu(dx)$ for every continuous real f that satisfies $\lim_{|x|\to\infty} f(x) = 0$.

  The above lemma proves that

$$\varphi_k(t) \stackrel{k\to\infty}{\longrightarrow} \varphi(t),$$

where

$$\varphi_k(t) = \exp\left\{\int_\Re \left(e^{itx} - 1 - itx\right) \frac{1}{x^2}\mu_k(dx)\right\}$$

and

$$\varphi(t) = \exp\left\{\int_{\Re} \left(e^{itx} - 1 - itx\right) \frac{1}{x^2}\mu(dx)\right\}.$$

Now you should see the reason why we put $1/x^2$ inside the integrand, because we require $f(x) = \left(e^{itx} - 1 - itx\right)\frac{1}{x^2} \to 0$ as $|x| \to \infty$.

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Corollary 1 (cf. Slide 26-52) Suppose a sequence of characteristic functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ has limits in every t, namely $\lim_{n\to\infty} \varphi_n(t)$ exists for every t. Define

$$g(t) = \lim_{n\to\infty} \varphi_n(t).$$

Then if $g(t)$ is continuous at $t = 0$, then there exists a probability measure μ such that

$$\mu_n \Rightarrow \mu, \text{ and } \mu \text{ has characteristic function } g$$

where μ_n is the probability measure corresponding to characteristic function $\varphi_n(\cdot)$.

As seen from Examples 28.1 and 28.2, a single-point-mass finite measure, either at $x = 0$ or at $x \neq 0$, leads to a characteristic function of some random variable, and its second moment is equal to its measure value on the point.

Infinitely divisible distribution

A multiple-point-mass finite measure can be represented as sum of single-point-mass finite measures; hence, the resultant $\varphi_k(t)$ is a product of many characteristic functions, and is itself a characteristic function. The second moment of $\varphi_k(t)$ is therefore the sum of the second moments of individual characteristic functions.

The limit $\varphi(t)$ of $\varphi_k(t)$ is apparently continuous; thus, $\varphi(t)$ is a **characteristic function** for some probability measure.

Infinitely divisible distribution

(Proof of the random variable corresponding to characteristic function $\varphi(t)$ having mean zero and variance $\mu(\Re)$.)

- **Theorem 25.11** If $X_n \Rightarrow X$, then

$$E[|X|] \le \liminf_{n\to\infty} E[|X_n|].$$

Examples 28.1 and 28.2 give that for point-mass measure μ_k, the corresponding variable has mean zero and second moment $E[X_k^2] = \mu_k(\Re)$. Hence, $E[X^2] \le \liminf_{k\to\infty} E[X_k^2] \le \sup_{k\ge 1} \mu_k(\Re) \underbrace{\le}_{\text{by definition of } \mu_k} \mu(\Re) < \infty$.

At this moment, we know the second moment of variable X corresponding to the limiting characteristic function $\varphi(\cdot)$ is finite. But, we still not yet know the values of its mean and second moment.

Lemma If $E[|X^n|] < \infty$, then

$$\varphi^{(n)}(0) = i^n E[X^n].$$

So we can take the first and second derivatives of $\varphi(t)$ to obtain:

$$\begin{aligned} i\cdot\text{mean} &= \varphi'(0)\\ &= \left(\int_{\Re}\left((ix)e^{itx}-ix\right)\frac{1}{x^2}\mu(dx)\right)\exp\left\{\int_{\Re}\left(e^{itx}-1-itx\right)\frac{1}{x^2}\mu(dx)\right\}\Bigg|_{t=0}\\ &= 0 \end{aligned}$$

and

$$\begin{aligned} i^2\cdot(\text{2nd moment}) &= \varphi''(0)\\ &= \left(\int_{\Re}\left((ix)^2e^{itx}\right)\frac{1}{x^2}\mu(dx)\right)\exp\left\{\int_{\Re}\left(e^{itx}-1-itx\right)\frac{1}{x^2}\mu(dx)\right\}\Bigg|_{t=0}\\ &+ \left(\int_{\Re}\left((ix)e^{itx}-ix\right)\frac{1}{x^2}\mu(dx)\right)^2\exp\left\{\int_{\Re}\left(e^{itx}-1-itx\right)\frac{1}{x^2}\mu(dx)\right\}\Bigg|_{t=0}\\ &= -\int_{\Re}\mu(dx) = -\mu(\Re). \end{aligned}$$

So $\varphi(t)$ corresponds to a distribution with mean 0 and finite variance $\mu(\Re)$.

Infinitely divisible distribution

(Proof of divisibility)

- Now let
$$\psi_n(t) = \exp\left\{\int_{\Re}\left(e^{itx} - 1 - itx\right)\frac{1}{x^2}\mu_n(dx)\right\}$$
where $\mu_n = \mu/n$.
Then $\varphi(t) = [\psi_n(t)]^n$, which implies that the distribution corresponding to $\varphi(t)$ is indeed infinitely divisible. $\square$

Theorem 28.2 Every infinitely divisible distribution with mean 0 and finite variance is the limit law of $S_n = X_{n,1} + \cdots + X_{n,r_n}$ for some independent triangular array satisfying:

1. $E[X_{n,k}] = 0$;
2. $\lim_{n\to\infty} \max_{1\leq k\leq r_n} E[X_{n,k}^2] = 0$;
3. $\sup_{n\geq 1} s_n^2 < \infty$, where $s_n^2 = \sum_{k=1}^{r_n} E[X_{n,k}^2]$.

Proof of Theorem 28.2

- *Claim:* If $X \perp\!\!\!\perp Y$ and $E[(X+Y)^2] < \infty$, then $E[X^2] < \infty$ and $E[Y^2] < \infty$.

  Proof: For any x, $|Y| \leq |x| + |x+Y|$ implies $E[|Y|] \leq |x| + E[|x+Y|]$.
  Hence, if $E[|Y|] = \infty$, then $E[|x+Y|] = \infty$ for every x,
  which implies $E[|X+Y|] = \infty$, a contradiciton to $E[(X+Y)^2] < \infty$.

  We can similarly prove that $E[|X|] < \infty$.

  Hence, by $x^2 + y^2 \leq (x+y)^2 + 2|x||y|$, we obtain

$$E[X^2] + E[Y^2] \leq E[(X+Y)^2] + 2E[|X|]E[|Y|] < \infty.$$

■

Infinitely divisible distribution

- Now suppose F is a cdf corresponding to an infinitely divisible distribution with mean 0 and variance $\sigma^2 < \infty$.

  If F is the n-fold convolution of F_n, then, by the previous claim, F_n must have finite mean and variance.

  Under "finiteness", We can then (safely) induce that:

  1. as n multiplying the variance of F_n is the variance of F, F_n has finite variance σ^2/n;
  2. as n multiplying the mean of F_n is the mean of F, F_n has mean 0.

  Take $r_n = n$ and $X_{n,1}, \ldots, X_{n,n}$ be i.i.d. with distribution F_n.
  Then

$$E[X_{n,k}] = 0, \quad \max_{1 \le k \le n} E[X_{n,k}^2] = \frac{\sigma^2}{n} \to 0, \quad \text{and} \quad s_n^2 = \sum_{k=1}^{n} \frac{\sigma^2}{n} = \sigma^2 < \infty.$$

  Consequently, Properties 1, 2 and 3 hold. □

Converse to Theorem 28.1

Theorem 28.1 For any finite measure (not necessarily probability measure) μ,

$$\varphi(t) = \exp\left\{ \int_{\Re} \left(e^{itx} - 1 - itx\right) \frac{1}{x^2} \mu(dx) \right\}$$

is the characteristic function of an infinitely divisible distribution with mean 0 and variance $\mu(\Re)$.

Theorem 28.3 If F is the limit law of $S_n = X_{n,1} + \cdots + X_{n,r_n}$ for an independent triangular array satisfying:

1. $E[X_{n,k}] = 0$;
2. $\lim_{n\to\infty} \max_{1\le k\le r_n} E[X_{n,k}^2] = 0$;
3. $\sup_{n\ge 1} s_n^2 < \infty$, where $s_n^2 = \sum_{k=1}^{r_n} E[X_{n,k}^2]$,

then F has characteristic function of the form

$$\varphi(t) = \exp\left\{ \int_{\Re} \left(e^{itx} - 1 - itx\right) \frac{1}{x^2} \mu(dx) \right\}$$

for some finite measure μ.

Converse to Theorem 28.1

In summary of Theorems 28.1, 28.2 and 28.3, for an independent triangular array satisfying:

1. $E[X_{n,k}] = 0$;
2. $\lim_{n\to\infty} \max_{1\le k\le r_n} E[X_{n,k}^2] = 0$;
3. $\sup_{n\ge 1} s_n^2 < \infty$, where $s_n^2 = \sum_{k=1}^{r_n} E[X_{n,k}^2]$,

F is the limit law of $S_n = X_{n,1} + \cdots + X_{n,r_n}$ if, and only if, F has characteristic function of the form

$$\varphi(t) = \exp\left\{ \int_{\Re} \left(e^{itx} - 1 - itx\right) \frac{1}{x^2} \mu(dx) \right\}$$

for some finite measure μ.

Proof of Theorem 28.3

- Let $\varphi_{X_{n,k}}(t)$ be the characteristic function of $X_{n,k}$.
  Let $\theta_{n,k}(t) = \varphi_{X_{n,k}}(t) - 1$.

- Since $E[X_{n,k}] = 0$,

$$|\theta_{n,k}(t)| = \left|\varphi_{X_{n,k}}(t) - 1\right| \le \frac{1}{2} t^2 E[X_{n,k}^2].$$

  Hence, Properties 2. and 3. respectively imply:

$$\lim_{n\to\infty} \max_{1\le k\le r_n} |\theta_{n,k}(t)| \le \frac{1}{2} t^2 \lim_{n\to\infty} \max_{1\le k\le r_n} E[X_{n,k}^2] = 0$$

  and

$$\sup_{n\ge 1} \sum_{k=1}^{r_n} |\theta_{n,k}(t)| \le \frac{1}{2} t^2 \sup_{n\ge 1} \sum_{k=1}^{r_n} E[X_{n,k}^2] < \infty.$$

Converse to Theorem 28.1

- Observe that

$$
\begin{aligned}
\left|\prod_{k=1}^{r_n}\varphi_{X_{n,k}}(t)-\exp\left\{\sum_{k=1}^{r_n}\left(\varphi_{X_{n,k}}(t)-1\right)\right\}\right| &\leq \sum_{k=1}^{r_n}\left|\varphi_{X_{n,k}}(t)-\exp\left\{\varphi_{X_{n,k}}(t)-1\right\}\right| \\
&= \sum_{k=1}^{r_n}\left|1+\theta_{n,k}(t)-e^{\theta_{n,k}(t)}\right| \\
&\leq \sum_{k=1}^{r_n}|\theta_{n,k}(t)|^2 e^{|\theta_{n,k}(t)|} \\
&\leq e^{t^2 s_n^2/2}\sum_{k=1}^{r_n}|\theta_{n,k}(t)|^2 \\
&\leq e^{t^2 s_n^2/2}\left(\max_{1\leq k\leq r_n}|\theta_{n,k}(t)|\right)\sum_{k=1}^{r_n}|\theta_{n,k}(t)| \\
&\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

$|\varphi_{X_{n,k}}(t)|<1$ implies that $(\varphi_{X_{n,k}}-1)=-a+jb$ for some $a>0$. Hence, $\left|\exp\left(\varphi_{X_{n,k}}-1\right)\right|=e^{-a}<1$

For complex z, $|e^z-1-z|\leq |z|^2\sum_{k=2}^{\infty}\frac{|z|^{k-2}}{k!}\leq |z|^2\sum_{k=2}^{\infty}\frac{|z|^{k-2}}{(k-2)!}=|z|^2 e^{|z|}$.

Converse to Theorem 28.1

Hence,

$$\varphi(t) = \lim_{n\to\infty} \varphi_{S_n}(t) = \lim_{n\to\infty} \exp\left\{\sum_{k=1}^{r_n}\left(\varphi_{X_{n,k}}(t)-1\right)\right\}.$$

- Denote by $F_{n,k}$ the cdf of $X_{n,k}$, then

$$\begin{aligned}
\sum_{k=1}^{r_n}\left(\varphi_{X_{n,k}}(t)-1\right) &= \sum_{k=1}^{r_n}\int_{\Re}(e^{itx}-1)dF_{n,k}(x) \\
&= \sum_{k=1}^{r_n}\int_{\Re}(e^{itx}-1-itx)dF_{n,k}(x), \quad (\text{by } E[X_{n,k}]=0) \\
&= \int_{\Re}(e^{itx}-1-itx)\sum_{k=1}^{r_n} dF_{n,k}(x) \\
&= \int_{\Re}(e^{itx}-1-itx)\frac{1}{x^2}\mu_n(dx),
\end{aligned}$$

where

$$\mu_n(-\infty, x] = \int_{-\infty}^{x}\mu_n(dy) = \int_{-\infty}^{x}\sum_{k=1}^{r_n} y^2 dF_{n,k}(y).$$

Notably, $\mu_n(\Re) = \sum_{k=1}^{r_n} E[X_{n,k}^2] = s_n^2$ is uniformly bounded in n. So, μ_n is a finite measure.

Converse to Theorem 28.1

- **Theorem 25.9 (Helly's theorem)** For every sequence $\{F_n\}_{n=1}^{\infty}$ of distribution functions, there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ and a non-decreasing, right-continuous function F (not necessarily a cdf) such that

$$\lim_{k\to\infty} F_{n_k}(x) = F(x)$$

for every continuous points of F.

Lemma Suppose that $\mu_n \stackrel{v}{\longrightarrow} \mu$ and $\sup_{n\geq 1} \mu_n(\Re) < \infty$. Then $\lim_{n\to\infty} \int_{\Re} f(x)\mu_n(dx) = \int_{\Re} f(x)\mu(dx)$ for every continuous real f that satisfies $\lim_{|x|\to\infty} f(x) = 0$.

Helly's theorem can be applied to finite measures as well; hence, there exists μ and subsequence $\{n_j\}_{j=1}^{\infty}$ such that μ_{n_j} converges to μ vaguely.

Theorem 25.9 (Helly's theorem for finite measures) For every sequence finite measure $\{\mu_n\}_{n=1}^{\infty}$, there exists a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \mu_{n_k}(-\infty, x] = \mu(-\infty, x]$$

for every continuous points of $\mu(-\infty, x]$.

Converse to Theorem 28.1

Since (by the 3rd assumption and the definition of μ_n on the bottom of Slide 28-20)

$$\sup_{j\geq 1}\mu_{n_j}(\Re) = \sup_{j\geq 1} s_{n_j}^2 \leq \sup_{n\geq 1} s_n^2 < \infty$$

and

$$\lim_{|x|\to\infty}\left|e^{itx}-1-itx\right|\frac{1}{x^2} \leq \lim_{|x|\to\infty}\min\left\{\frac{|x|^2}{2!},\frac{|x|}{1!}\right\}\frac{1}{x^2} = \lim_{|x|\to\infty}\min\left\{\frac{1}{2},\frac{1}{|x|}\right\} = 0,$$

we obtain

$$\lim_{j\to\infty}\int_{\Re}(e^{itx}-1-itx)\frac{1}{x^2}\mu_{n_j}(dx) = \int_{\Re}(e^{itx}-1-itx)\frac{1}{x^2}\mu(dx).$$

- But,

$$\lim_{n\to\infty}\int_{\Re}(e^{itx}-1-itx)\frac{1}{x^2}\mu_n(dx) = \varphi(t).$$

Consequently,

$$\varphi(t) = \int_{\Re}(e^{itx}-1-itx)\frac{1}{x^2}\mu(dx)$$

for the vague limit μ of μ_{n_j}, and $\mu(\Re) \leq \sup_n \mu_n(\Re) < \infty$ is a finite measure.

□

Example of limit law

Double exponential: $\boxed{\text{pdf} \equiv \frac{1}{2}e^{-|x|} \text{ for } -\infty < x < \infty, \text{ and } \varphi(t) = \frac{1}{1+t^2}.}$

Define $\mu(-\infty, x] = \int_{-\infty}^{x} |y|e^{-|y|}dy$.

Hence, $\mu(\Re) = \int_{-\infty}^{\infty} |y|e^{-|y|}dy = 2$, and

$$
\begin{aligned}
&\exp\left\{\int_{-\infty}^{\infty}(e^{itx} - 1 - itx)\frac{1}{x^2}|x|e^{-|x|}dx\right\} \\
&= \exp\left\{\int_{-\infty}^{\infty}(e^{itx} - 1 - itx)\frac{1}{|x|}e^{-|x|}dx\right\} \\
&= \exp\left\{\int_{-\infty}^{0}(e^{itx} - 1 - itx)\frac{1}{(-x)}e^{x}dx + \int_{0}^{\infty}(e^{itx} - 1 - itx)\frac{1}{x}e^{-x}dx\right\} \\
&= \exp\left\{\int_{0}^{\infty}(e^{-itx} - 1 + itx)\frac{1}{x}e^{-x}dx + \int_{0}^{\infty}(e^{itx} - 1 - itx)\frac{1}{x}e^{-x}dx\right\} \\
&= \exp\left\{\int_{0}^{\infty}\frac{2[\cos(tx) - 1]}{x}e^{-x}dx\right\} \\
&= \exp\{-\log(1+t^2)\} = \frac{1}{1+t^2}.
\end{aligned}
$$

Example of limit law

centered exponential density:

Let random variable Y have pdf e^{-y} for $0 \le y < \infty$.

$$\int_0^\infty e^{ity} e^{-y} dy \;=\; \frac{1}{1-it}.$$

Let $X = Y - 1$.

Hence, $E[e^{itX}] = E[e^{it(Y-1)}] = \dfrac{e^{-it}}{1-it}$.

Let $\mu(dx) = xe^{-x}$ for $0 \le x < \infty$.

Then $\mu(\Re) = \displaystyle\int_0^\infty xe^{-x} dx = 1 < \infty$, and

$$\begin{aligned}
\exp\left\{\int_0^\infty (e^{itx} - 1 - itx)\frac{1}{x^2} xe^{-x} dx\right\} &= \exp\left\{\int_0^\infty (e^{itx} - 1 - itx)\frac{1}{x} e^{-x} dx\right\} \\
&= \exp\left\{\int_0^\infty (e^{itx} - 1)\frac{e^{-x}}{x} dx\right\} \exp\left\{-it \int_0^\infty e^{-x} dx\right\} \\
&= \frac{e^{-it}}{1-it}
\end{aligned}$$

Example of limit law

- **centered gamma distribution** is also infinitely divisible with $\mu(dx) = uxe^{-x}$ for $0 < x < \infty$.

- **Cauchy distribution** is an infinitely divisible distribution with **infinite** second moment. (So its canonical formula is a little different from the one shown in Theorem 28.1.)

$$\begin{aligned}
&\exp\left\{\int_{-\infty}^{\infty}\left(e^{itx}-1\right)\mu(dx)\right\}\exp\left\{\int_{-\infty}^{\infty}\left(e^{itx}-1-itx\right)\frac{1}{x^2}\mu(dx)\right\}\\
&= \exp\left\{\int_{-\infty}^{\infty}\left(e^{itx}-1\right)\mu(dx)+\int_{-\infty}^{\infty}\left(e^{itx}-1-itx\right)\frac{1}{x^2}\mu(dx)\right\}\\
&= \exp\left\{\int_{-\infty}^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\mu(dx)\right\}\\
&= e^{-|t|},
\end{aligned}$$

where $\mu(dx) = \dfrac{dx}{\pi(1+x^2)}$.

- The product of infinitely divisible characteristic functions is also an infinitely divisible characteristic function.