Introduction to Combinatorics Lecture 8

An Introduction of Extremal Set Theory

Research Problem. Under a constraint or a collection of constraints, find the maximum number of sets satisfying the given constraints.
Clearly, the collection of sets, $\mathbb{B}$, from $\mathbb{X}$ is also a design $(\mathbb{X}, \mathbb{B})$.

Notation.

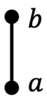
- $[n] = \{1, 2, ..., n\}$.
- $\binom{[n]}{k}$ $=_{def}$ the collection of (all) k-subsets of $[n]$.
- $\binom{n}{k} = |\binom{[n]}{k}|$.

Definition 8.1 (Partial ordered set)**.** $\mathbb{X} = \{x_1, x_2, ..., x_n\}$ is a set of n elements and '$\leq$' is a partial order defined on $\mathbb{X}$. $\langle \mathbb{X}, \leq \rangle$ is called a partial order set, Poset in short.

Definition 8.2 (Partial order)**.** '$\leq$' is a partial order of $\mathbb{X}$ if

1. Reflexivity: $a \leq a \quad \forall a \in \mathbb{X}$
2. Anti-symmetry: $a \leq b$ and $b \leq a$ imply $a = b \quad \forall a, b \in \mathbb{X}$, and
3. Transitivity: $a \leq b, \ b \leq c$ imply $a \leq c \quad \forall a, b, c \in \mathbb{X}$.

Definition 8.3 (Total order)**.** '$\leq$' is a total order of Y provided any two distinct elements in Y, y_i and y_j, either $y_i \leq y_j$ or $y_j \leq y_i$. (y_i and y_j are comparable.)

We may use a graph to depict a partial ordered set (Poset), $\langle S, \leq \rangle$. It is known as the Hasse-diagram. Mainly, if $a, b \in S$ and $a \leq b$, then the vertex representing b is higher than a as: 

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For example, $\langle 2^{[4]}, \subseteq \rangle$ can be represented as follows.

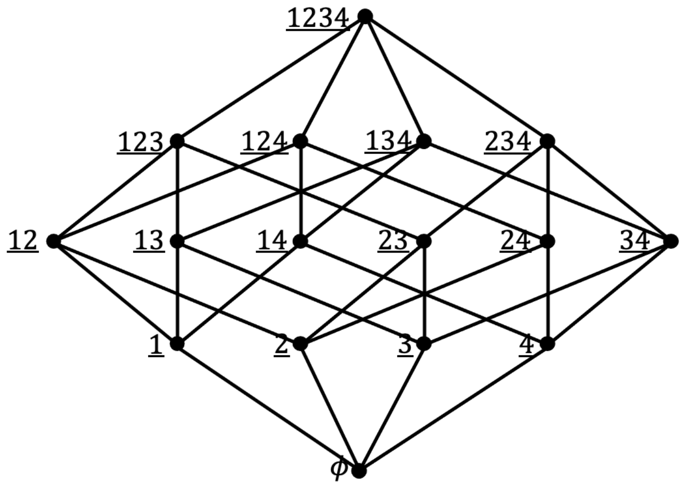


Figure 8.1: Hasse-diagram of $\langle 2^{[4]}, \subseteq \rangle$.

For convenience, this diagram can be considered as a graph (in Figure 8.2) and only the structure will be studied.

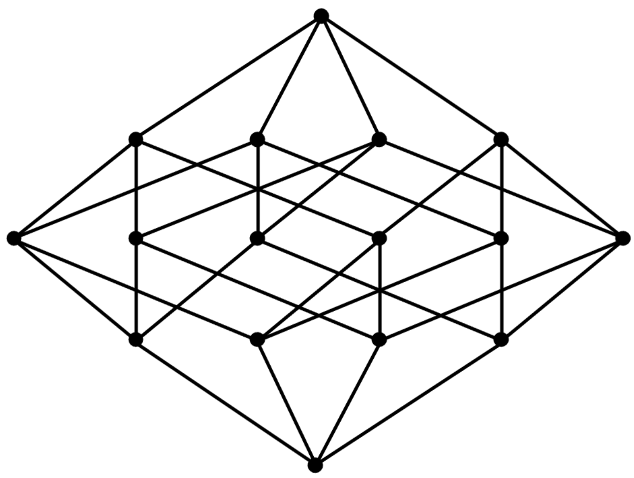


Figure 8.2

Definition 8.4 (Anti-chain, Chain)**.** A subset of a poset in which no two distinct elements are comparable is called an anti-chain. On the other hand, a totally ordered set is called a chain.

Example. The blue vertices are an anti-chain and the orange path is a chain.

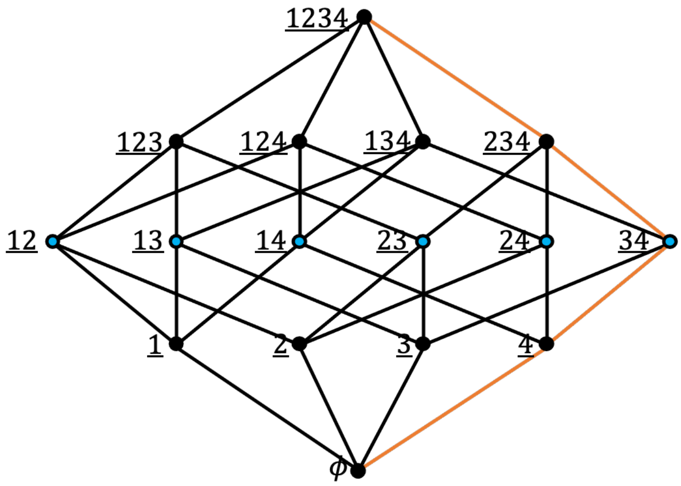
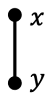
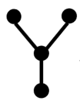


Figure 8.3: Poset with set-containment.

Extremal set problem.

Given a configuration of posets, say $I = P_2$: 

($y \leq x$), find the maximum number of sets in $2^{[n]}$ such that the induced partial ordered set contains no sub-poset which is given, i.e., contains no P_2.

We can change $I = P_2$ to any kinds of sub-poset. For example, P_3 :  or S_3 (star of order 3):  . T he result solving case $I = P_2$ is known as the Sperner's theorem.

Theorem 8.1 (Sperner's theorem)**.** *Consider the collection of all subsets of* $[n]$*. The maximum number of subsets which do not contain each other is equal to* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$*. (The maximum anti-chain problem.)*

Proof. Let $\mathbb{B}$ be a collection of subsets which do not contain each other and attains the maximum. Furthermore, let a_k be the number of sets in $\mathbb{B}$ whose size is k. Hence, $|\mathbb{B}| = \sum_{k=0}^{n} a_k$. Note that a_i's may be zero. Since $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ is clearly an anti-chain, $|\mathbb{B}| \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. So, it is suffices to prove $|\mathbb{B}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

<u>Claim (Lubell-Yamamoto-Meshalkin, LYM inequality)</u>. $\sum_{k=0}^{n} a_k / \binom{n}{k} \leq 1$.

Consider the set of permutations of $[n]$. Clearly, there are $n!$ permutations. Now, for each set $S = \{s_1, s_2, ..., s_k\}$ in $\mathbb{B}$, we associate this set with $|S|!(n-|S|)!$ permutations by taking the maximum chain passing $s_1 s_2 \cdots s_k$. ($\emptyset - s_1' - s_1's_2' - s_1's_2's_3' - \cdots - s_1 s_2 \cdots s_k - s_1 s_2 \cdots s_k s_{k+1}' - \cdots - [n]$ where $s_i' \in \{s_1, s_2, ..., s_k\}$ for $1 \le i \le k$.) Note that each permutation can only be associated with a single set in $\mathbb{B}$. Two sets in $\mathbb{B}$ do not contain each other. Now we have

$$\sum_{S\in\mathbb{B}} |S|!(n-|S|)! = \sum_{k=0}^{n} a_k \cdot k!(n-k)! \le n!$$

Hence, $\displaystyle\sum_{k=0}^{n} a_k \cdot \frac{k!(n-k)!}{n!} \le 1$.

Since $1 \ge \displaystyle\sum_{k=0}^{n} a_k / \binom{n}{k} \ge \sum_{k=0}^{n} a_k / \binom{n}{\lfloor \frac{n}{2} \rfloor}$, $\binom{n}{\lfloor \frac{n}{2} \rfloor} \ge \sum_{k=0}^{n} a_k = |\mathbb{B}|$. The proof follows. □

Example. $n = 5$.

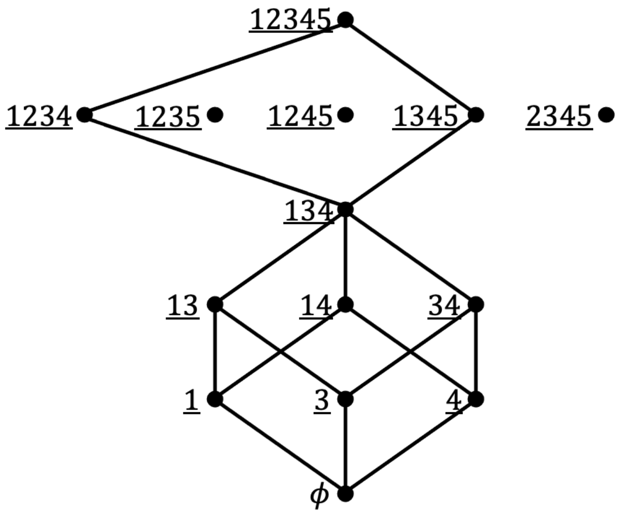


Figure 8.4: $(5-3)! \cdot 3!$ maximum chains.

Problem. Find the maximum number of subsets in $2^{[n]}$ such that their induced poset does not contain P_3. A good guess is $\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$. But is it true? Try it!

Problem. Find the maximum collection of sets $B_{n,r}$ of size r which are mutually intersection, that is, $\forall S_1, S_2 \in B_{n,r},\ S_1 \cap S_2 \neq \emptyset$. $B_{n,r}$ is called an r-uniform intersection family defined on $[n]$. The following theorem is a beautiful result of this problem.

Theorem 8.2 (Erdös-Ko-Rado, EKR theorem)**.** $|B_{n,r}| = \binom{n-1}{r-1} \quad n \in \mathbb{N}$.

Proof. Let $B = \{S \cup \{n\} \mid S \in \binom{[n-1]}{r-1}\}$. Then, B is an intersection family of $[n]$ since each set contains the element n. Hence, $|B_{n,r}| \geq \binom{n-1}{r-1}$. Next, we prove that $|B_{n,r}| \leq \binom{n-1}{r-1}$.

Observe that if we let $(a_1, a_2, ..., a_n)$ be a cyclic permutation of $[n]$, then this cycle contains at most r sets of $B_{n,r}$. For example, when $n = 8$ and $r = 3$, let $(3, 1, 8, 2, 7, 5, 6, 4)$ be an arbitrary cyclic permutation. Now, if $\{8, 2, 7\} \in B_{8,3}$, then we have two more possible sets $\{1, 8, 2\}$ and $\{2, 7, 4\}$. So, for general n, we have at most $r \cdot (n-1)!$ sets for intersecting family. By the same idea in Sperner's theorem, each set in $B_{n,r}$ can be associated with $r!(n-r)!$ permutations. Hence, $|B_{n,r}| \cdot r!(n-r)! \leq r \cdot (n-1)!$. Therefore, $|B_{n,r}| \leq \dfrac{(n-1)!}{(r-1)!(n-r)!} = \binom{n-1}{r-1}$. □

Example. $|B_{7,3}| = \binom{6}{2} = 15$.

Another good problem to study related to sets.

Let $n = 2t + 1$. We may define a graph G as follows: $V(G) = \binom{[n]}{t}$ and two vertices are adjacent if and only if their intersection is an empty set. The graph G is known as an *odd* graph of order n, denoted by O_n.

Example. O_5 $(n = 5,\ t = 2)$. It is in fact the Petersen graph.

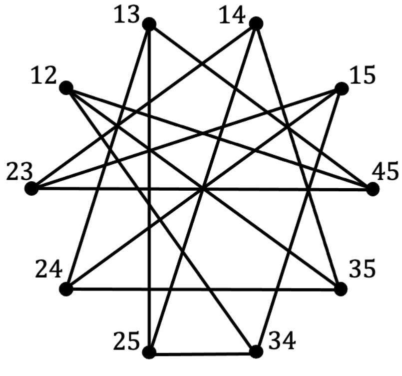


Figure 8.5: O_5.

Study the structure of O_n is an important problem in both Graph Theory and Design Theory.

If we further require that any two r-sets in $2^{[n]}$ can have at most one element in common, thus exactly one element in common, then the collection of such r-sets, denoted by $B_{n,r}^{(1)}$ has at most $\dfrac{n(n-1)}{r(r-1)}$ sets.
To see this, we notice that any pair of elements in $[n]$ can occur in at most one r-set of $B_{n,r}^{(1)}$. Hence, the pairs we have in total is $\dfrac{n(n-1)}{2} = \binom{n}{2}$ and each r-set can use $\binom{r}{2} = \dfrac{r(r-1)}{2}$ pairs, this implies that $|B_{n,r}^{(1)}| \leq \binom{n}{2}/\binom{r}{2}$.

For some n and r, the equality does hold. For example, $B_{7,3}^{(1)} = \{124, 235, 346, 457, 672, 713\}$ (Fano plane), and $B_{13,4}^{(1)} = \{\{0,1,3,9\} + i \mid i \in \mathbb{Z}_{13}\}$ ($|B_{13,4}^{(1)}| = \dfrac{13 \times 12}{4 \times 3} = 13$).

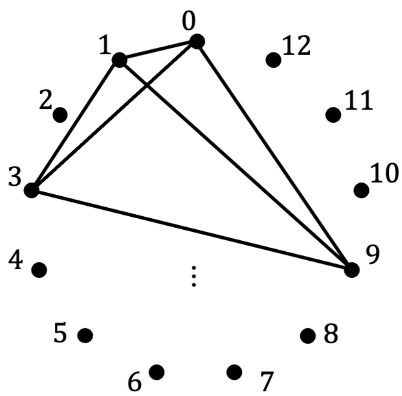


Figure 8.6: $B_{13,4}^{(1)}$

Block Design

The study of the incidence structures between finite sets is one of the most important topics in Combinatorial Theory. There are three basic directions: (1) Finite Geometry, (2) Block Design, and (3) Hypergraph. It is not easy to describe the difference between them. In general, 'Finite Geometry' cares more about the property related to the geometry on a plane, 'Block Design' emphasizes on numerical relationship, and 'Hypergraph' focuses on arbitrarily given edges (finite subsets).

Therefore, to study Block Design, we start with the construction of designs of small order. We also find the necessary conditions for the existence of the kind of designs we would like to obtain. Following that, we then put forth to prove the necessary conditions are also sufficient by constructing all such designs. In general, the part on necessary conditions is comparatively easier. As to construction part, some of the design does not exist even we know the necessary conditions. We shall see that in next section.

Definition 8.5 (Block design)**.** $(\mathbb{X}, \mathbb{B})$ is a design if $\mathbb{X}$ is a non-empty set and $\mathbb{B}$ is a collection of subsets of $\mathbb{X}$. If all the subsets are of the same cardinality, then $(\mathbb{X}, \mathbb{B})$ is called a block design. For convenience, all the sets in $\mathbb{B}$ are referred as blocks in $\mathbb{X}$.

Definition 8.6 (Simple design)**.** If all the subsets of a design $(\mathbb{X}, \mathbb{B})$ are all distinct, then it is a simple design. Note that $\mathbb{B}$ can be a multi-set in a design, the blocks with repeated occurrence is known as repeated blocks.

Definition 8.7 (Representation of design)**.** Let $\mathbb{X} = \{x_1, x_2, ..., x_v\}$ be the set of 'varieties' and $\mathbb{B} = \{B_1, B_2, ..., B_b\}$ be the set of blocks. Then, we can define a variety-block incidence matrix to represent the design, say **A**, and also a bipartite graph to represent $(\mathbb{X}, \mathbb{B})$, say $\mathbf{G}_{\mathbb{X},\mathbb{B}}$. $A = \Big[a_{i,j}\Big]_{v\times b}$ where $a_{i,j} = \begin{cases} 1 & \text{if } x_i \in B_j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$

Therefore, **A** is a $(0, 1)$-matrix.

Definition 8.8 (Pairwise balanced design, PBD)**.** An $(\mathbb{X}, \mathbb{B})$ is called a pairwise balanced design (PBD for short) if for any pair of elements in $\binom{\mathbb{X}}{2}$, they occur together in exactly λ blocks of $\mathbb{B}$. Notice that in PBD, the blocks are not necessarily be of the same. So, it is denoted by $2-(v, K, \lambda)$ design where $|\mathbb{X}| = v$.

Example.

1. A $2-(6, \{2, 5\}, 1)$ design: $\mathbb{X} = \mathbb{Z}_6$ and $\mathbb{B} = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}, \{1, 2, 3, 4, 5\}\}$.
2. $\mathbb{X} = \mathbb{Z}_v$ and $\mathbb{B} = \binom{\mathbb{Z}_v}{k}$, $k \geq 2$. Then, $(\mathbb{X}, \mathbb{B})$ is a $2-(v, k, \lambda)$ design where $\lambda = \dfrac{r(k-1)}{v-1}$.
   Note that $r = \binom{v-1}{k-1} = \dfrac{(v-1)!}{(k-1)!(v-k)!} = \dfrac{(v-1)(v-2)\cdots(v-k+1)}{(k-1)!}$,
   hence $\lambda = \dfrac{(v-1)(v-2)\cdots(v-k+1)}{(k-1)!} \cdot \dfrac{k-1}{v-1} = \binom{v-2}{k-2}$.

Remark. $(\mathbb{X}, \mathbb{B})$ is also a $t-(v, k, \lambda)$ design for all $2 \leq t \leq k < v$.

The following notions are not related to vector spaces.

Definition 8.9 (Partial linear space, Linear space)**.** An $(\mathbb{X}, \mathbb{B})$ is called a partial linear space if any two blocks of $\mathbb{B}$ contains at most one common element. If, indeed, any two elements (varieties) of a partial linear space occur together in a block of $\mathbb{B}$, then $(\mathbb{X}, \mathbb{B})$ is a linear space with index 1.

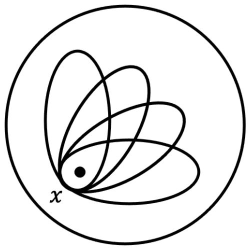
Remark. We can use 'Geometry' to refer the above definitions:

- Partial linear space: Any two lines intersect at most one point.
- Linear space: Any two points lie on a line (some line) of a partial linear space.

Basic properties of a design.

1. If $(\mathbb{X}, \mathbb{B})$ is a $2-(v, k, \lambda)$ design, then we have
   (a) for each $x \in \mathbb{X}$, $r_x = r = \dfrac{\lambda(v-1)}{k-1}$ or $r(k-1) = \lambda(v-1)$.
   (b) $b = |\mathbb{B}| = \dfrac{\lambda v(v-1)}{k(k-1)}$ or $bk = rv$.

Proof. Since x occurs with (each of) all the other $v-1$ elements exactly in λ blocks, r_x is equal to $\lambda(v-1)$ possible such pairs divided by the $k-1$ pairs which can be obtained from a block. (The second equality is a consequence of the above idea by using two-way counting.) This concludes the proof of (a).



As to (b), it is a direct counting of the number of pairs occur in $\mathbb{B}$ via the number of pairs occur in a block. Therefore $|\mathbb{B}| = \frac{\lambda\binom{v}{2}}{\binom{k}{2}}$. The second identity comes from the (total) occurrence of elements. $\square$

2. (Fisher's inequality) If $(\mathbb{X}, \mathbb{B})$ is a $2-(v,k,\lambda)$ design, then $|\mathbb{X}| \leq |\mathbb{B}|$.

Proof. Let A be the incident matrix of $(\mathbb{X}, \mathbb{B})$. Then, $AA^T = (r-\lambda)I + \lambda J$, i.e., AA^T is a $v \times v$ matrix such that each entry in the diagonal is r and each entry out side diagonal is λ.

$$AA^T = \begin{array}{c} \\ x_1 \\ \vdots \\ x_v \end{array} \overset{\begin{array}{cccc} B_1 & B_2 & \cdots & B_b \end{array}}{\left[\begin{array}{cccc} & & & \\ & & & \\ & & & \end{array} \right]_{v \times b}} \left[\begin{array}{c} \\ \\ \\ \\ \end{array} \right]_{b \times v}$$

Note that $AA^T(i,j)$ is the inner product of the ith row and the jth row. So, if $i=j$, it is the occurrence of x_i ($r_{x_i} = r$) in the blocks of $\mathbb{B}$, and if $i \neq j$, it is the number of blocks in which x_i and x_j occur together in the blocks, λ.

Now, we can find $det(AA^T) = k \cdot r \cdot (r-\lambda)^{v-1}$. (Gaussian elimination.) Since $v > k$, $\lambda < r$. This concludes that AA^T is non-singular, i.e., $rank(AA^T) = v$. Furthermore, $rank(AA^T) \leq rank(A) \leq \min\{v, b\}$, hence $b \geq v$.

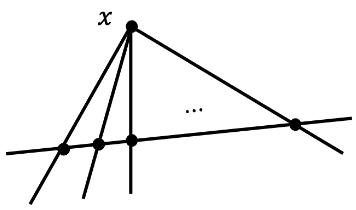
In what follows, we find $det(AA^T)$ by using its eigenvalues. Since $AA^T = (r-\lambda)I + \lambda J$, an eigenvalue μ satisfies $(AA^T)x = \mu x = (r-\lambda)x + \lambda Jx = (r-\lambda)x + \lambda\mu' x$ where

μ' is an eigenvalue of J. By the fact that J is of rank 1, the set of eigenvalues of J are $\{v, 0, 0, ..., 0\}$ (0 with multiplicity $v-1$). Hence, $\mu x = ((r-\lambda)+\lambda\mu')x$. This implies that $\mu = r-\lambda$ ($v-1$ of them) and $\mu = r-\lambda+\lambda v = r+\lambda(v-1) = r+(k-1)r = kr$. Thus, $det(AA^T) = k \cdot r \cdot (r-\lambda)^{v-1}$.

(Note here that using the spectrum of an adjacency matrix of a graph is one of the main subjects of **Algebraic Graph Theory**.) □

Theorem 8.3. *If* $(\mathbb{X}, \mathbb{B})$ *is a linear space, then* $|\mathbb{X}| \leq |\mathbb{B}|$.

Proof. Again, let $|\mathbb{X}| = \{x_1, x_2, ..., x_v\}$ and $\mathbb{B} = \{B_1, B_2, ..., B_b\}$. Since $(\mathbb{X}, \mathbb{B})$ is a linear space, any two elements in $\mathbb{X}$ occur together in a block of $\mathbb{B}$. Assume that $b \leq v$. Here is an important observation: If $x \notin B_i$, then $r_x \geq |B_i|$ since each element of B_i is going to occur together with x in some other blocks in $\mathbb{B}$.



Now, we are ready for the following statements.

$$1 = \sum_{B \in \mathbb{B}} \frac{1}{b} = \sum_{B \in \mathbb{B}} (\sum_{x \notin B} \frac{1}{b(v-|B|)}) \tag{1}$$

$$1 = \sum_{x \in \mathbb{X}} \frac{1}{v} = \sum_{x \in \mathbb{X}} (\sum_{B \not\ni x} \frac{1}{v(b-r_x)}) \tag{2}$$

$$vr_x \geq b|B| \text{ for each } x \notin B \quad (v \geq b). \tag{3}$$

By (1), (2) and (3),

$$\frac{1}{b} = \sum_{x \notin B} \frac{1}{b(v-|B|)} \leq \sum_{B \not\ni x} \frac{1}{v(b-r_x)} = \frac{1}{v}$$

we have $b \geq v$, a contradiction. Hence, $b \geq v$. □

Remark. The equality $v = b$ also shows that $r_x = |B|$ for each $x \in \mathbb{X}$ and $B \in \mathbb{B}$. The implication of this fact is that any two blocks intersect at exactly one element, i.e., $|B_i \cap B_j| = 1$ for all $1 \leq i \neq j \leq b$.

Definition 8.10 (Projective plane)**.** $(\mathbb{X}, \mathbb{B})$ is a projective plane if $(|\mathbb{X}| = |\mathbb{B}|)$ and $(\mathbb{X}, \mathbb{B})$ is a linear space.

Definition 8.11 (SBIBD)**.** A BIBD is a square BIBD, denoted by SBIBD, if $v = b$.

The following Theorem is well-known, we state it and omit the proof here. (It is a 'necessary condition' for the existence of an SBIBD.)

Theorem 8.4 (Bruck-Ryser-Chowla, 1949-1950)**.** *If a* $2-(v, k, \lambda)$ *design is a square BIBD, then*

1. $k - \lambda$ *is a square of an integer when* v *is even; and*
2. $z^2 = (k-\lambda)x^2 + (-1)^{\frac{v-1}{2}} \cdot \lambda y^2$ *has a nonzero integral solution when* v *is odd.*

Remark. (1) is easy to see: $det(AA^T) = det(A)^2 = kr(r-\lambda)^{v-1} = k^2(k-\lambda)^{v-1}$ ($v = b$ implies $r = k$). However, the proof of (2) is quite complicate, we omit it.

Special designs related to Geometry.

Definition 8.12 (Projective plane and Affine plane)**.** A Steiner 2-design $S(2, n+1, n^2+n+1)$ is called a projective plane of order n, denoted by $PG(2, n)$. A Steiner 2-design $S(2, n, n^2)$ is an affine plane of order n, denoted by $AG(2, n)$.

Facts.

1. The existence of a $PG(2, n)$ is 'equivalent' to the existence of an $AG(2, n)$.

   Proof. (More details will be given later.)

$$PG(2,n) \overset{\text{deleting a block}}{\Longrightarrow} AG(2,n)$$
$$AG(2,n) \overset{\text{adding a line at infinity}}{\Longrightarrow} PG(2,n)$$

□

2. A $PG(2, n)$ does exist for each n when n is a prime power.

3. No other kind of $PG(2, n)$ has been founded.
4. A $PG(2, n)$ does not exist for $n = 1, 2, 6, 10$ and possibly others.
5. We can extend $AG(2, n)$ and $PG(2, n)$ to $AG(d, n)$ and $PG(d, n)$ for $d \geq 3$ respectively. But, the constructions are getting harder.

Example.

1. $n = 2,\ AG(2, 2):\quad \mathbb{X} = \mathbb{Z}_4,\ \mathbb{B} = \{\underbrace{\underline{\{0,1\},\{2,3\}}, \underline{\{1,2\},\{0,3\}}, \underline{\{1,3\},\{0,2\}}}_{\text{parallel classes}}\}.$
2. $n = 2,\ PG(2, 2):\quad \mathbb{X} = \mathbb{Z}_7,\ \mathbb{B} = \{\{0,1,4\},\{2,3,4\},\{0,2,5\},\{1,3,5\},\{0,3,6\}, \{1,2,6\},\{4,5,6\}\}.$

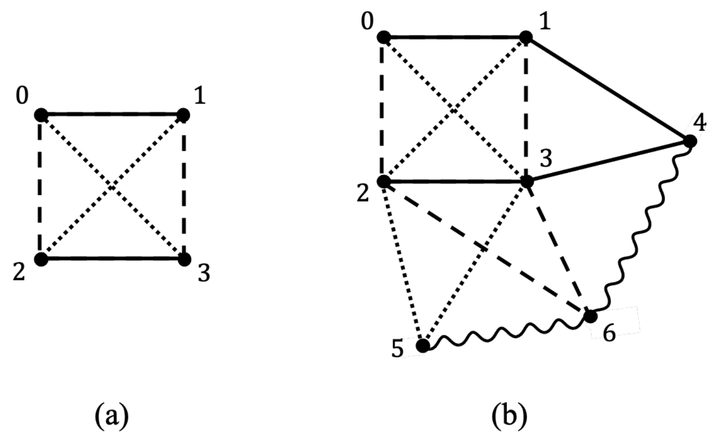


Figure 8.7: (a) $AG(2, 2)$. (b) $PG(2, 2)$.

Remark.

- A $PG(2, n)$ is a symmetric design, i.e., $|\mathbb{X}| = |\mathbb{B}|$.
- An $AG(2, n)$ contains parallel classes, each has n blocks. In fact, there are $n + 1$ parallel classes.
- A parallel class of a design is a collection of blocks $B_1, B_2, ..., B_t$ such that $\cup_{i=1}^{t} B_i = \mathbb{X}$.