§16.4 Green's Theorem (‘二維’的微積分基本定理)

＊ 名詞介紹：

一個區域 D 的邊界是 C positive orientation：沿著邊界 C 走的方向要使得 D 永遠是在 C 的左側

(i) 

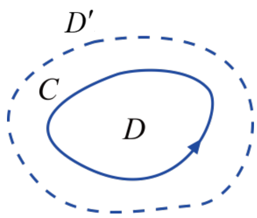
(counterclockwise 逆時鐘方向)

(ii) 

(外邊界是逆時鐘，裏邊界是順時鐘)

Green’s Theorem:

已知： (i)



C : positively oriented, piecewise smooth, simple closed curve.

(ii) $\mathbf{F} = \langle P, Q \rangle$： P, Q : continuous partial derivations on D'.

結論： $$\iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P dx + Q dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Remarks：

(i) 符號：$\int_C P dx + Q dy = \oint_C P dx + Q dy = \oint_C P dx + Q dy = \int_{\partial D} P dx + Q dy$

(ii) n 維區域 D 的邊界 ∂D 是 $n-1$ 維，微積分基本定理的精神之一就是一個函數的某種微分的 n 重積分作用在 D，等於原函數的 $(n-1)$ 重積分作用在 ∂D.

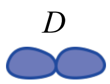
(iii) If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $\int_{\partial D} P\mathrm{dx} + Q\mathrm{dy} = \mathrm{A}(D)$.

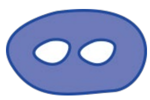
因此若選 (i) $Q = x, P = 0$; or
(ii) $Q = 0, P = -y$; or
(iii) $Q = \frac{1}{2}x, P = -\frac{1}{2}y$.

都可透過 line integral 來求面積.

(iv) **Theorem 16.3.3** (檢查保守場的充分條件) :
Green's Theorem + Theorem 16.3.2 - 16.3.4 .

(v) **可推廣**：若 D 是 (i) finite union of simple regions

如 

或(ii) region with holes 如  .

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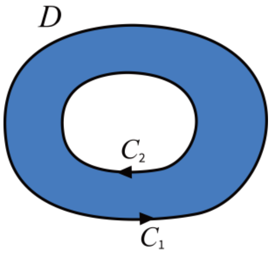
＊ 以此圖的 D 為例，中間有個洞(hole) 外邊界為 C_1，裏邊界為 C_2.



對上圖 Green's Theorem 仍然適用：$\mathbf{F}=\langle P,Q\rangle$.

(i) $\iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dA = \int_{C_1\cup C_2} \mathbf{F}\cdot d\mathbf{r}$，此處 C_1, C_2 都是正向,

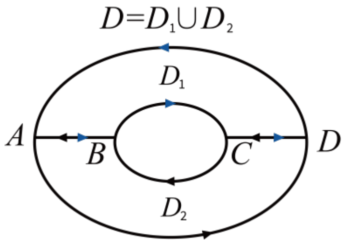
也即 C_1 是逆時鐘, C_2 是順時鐘.

(ii) if $\mathbf{F}$ is conservative, then $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = 0$, and, hence,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = -\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{-C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Here C_1 and C_2 都是逆時鐘.

Proof of (i)：將 D 分成 D_1 和 D_2 如右圖.
則 Green's Theorem 可分別用在 D_1 和 D_2,
其相對應的正向邊界(逆時鐘)分別為 ∂D_1 和 ∂D_2.



$$\iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dA = \iint_{D_1} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dA + \iint_{D_2} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dA$$

$$= \oint_{\partial D_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F}\cdot d\mathbf{r}$$

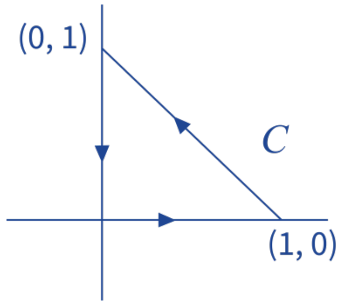
$$= \oint_{\partial D_1 \cup \partial D_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{C_1\cup C_2} \mathbf{F}\cdot d\mathbf{r}\ .\ \text{(why?)}$$

註：$\overline{AB}$ 和 $\overline{CD}$，正反方向皆有，互相抵消，剩下 $C_1\cup C_2$.

先算兩個例子，再回來證明定理

Example 1 $\int_C x^4 dx + xy\, dy$.



Solution : $\int_C x^4 dx + xy\, dy$

$$= \iint y\,dA = \int_0^1 \int_0^{1-y} y\, dx\, dy$$

$$= \int_0^1 \left(y - y^2\right) dy = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example 2 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 算橢圓面積.

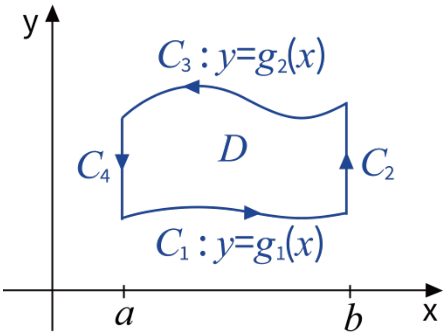
Solution : $\mathbf{F} = \left\langle \frac{-1}{2} y, \frac{1}{2} x \right\rangle = \langle P, Q \rangle$

$$A(D) = \int_{\partial D} P dx + Q dy \qquad x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$$

$$= \int_0^{2\pi} \left(-\left(-\frac{1}{2} b \sin\theta \right) a \sin\theta + \frac{1}{2} a (\cos\theta) b \cos\theta \right) d\theta$$

$$= \pi ab.$$

Proof : Special case : D : type I $\cap$ type II region = simple region



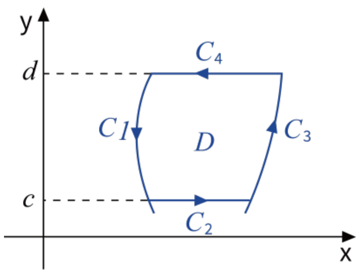
$$-\iint_D \frac{\partial P}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy\, dx$$

$$= -\int_a^b \left(P(x, g_2(x)) - P(x, g_1(x)) \right) dx$$

$$= \int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx$$

$$= \int_{C_1} P dx + \int_{C_2} P dx + \int_{C_3} P dx + \int_{C_4} P dx$$

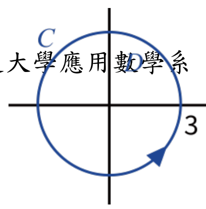
$$= \int_C P dx.$$



$C_1 : x = g_1(y)$

$C_3 : x = g_2(y)$

Similarly, $\iint_D \frac{\partial Q}{\partial x} dA = \int_C Q dy.$

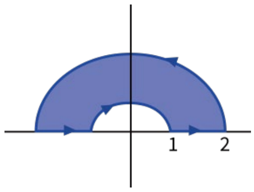


Example 3 Evaluate : $\int_C \underbrace{\left(3y - e^{\sin x}\right)}_{P} dx + \underbrace{\left(7x + \sqrt{y^4+1}\right)}_{Q} dy$

Solution : $\int_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$

$$= \iint_{x^2+y^2\le 3^2} (7-3)\, dA = 4\times 9\pi = 36\pi.$$

Example 4 Evaluate : $\int_C y^2 dx + 3xydy$



Solution : $\int_C y^2 dx + 3xydy$

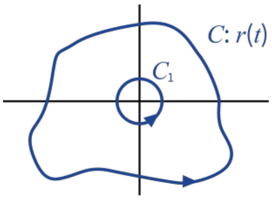
$$= \iint_D (3y - 2y)\, dA$$

$$= \int_0^\pi \int_1^2 (r\sin\theta) r dr d\theta = \int_0^\pi \sin\theta d\theta \int_1^2 r^2 dr$$

$$= 2\cdot\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{14}{3}.$$

Example 5 $\mathbf{F}(x,y)=\left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle$

Prove that $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ for any positively oriented simple closed path that encloses the origin.



Proof : (i) $\dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{\left(x^2+y^2\right)^2} = \dfrac{\partial Q}{\partial x}.$

(ii) By Green's Theorem, $0 = \int_{C\cup(-C_1)} \mathbf{F}\cdot d\mathbf{r}$.

(iii) 由(ii) 得

$$0 = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left(\frac{-a\sin t}{a^2} a(-\sin t) + \frac{a\cos t}{a^2} a\cos t \right) dt$$

$$= 2\pi.$$

Example 6 Evaluate : $\int_C y^3dx - x^3dy \quad C: x^2+y^2=4$

Solution : $\int_C y^3dx - x^3dy$

$$= \iint\limits_{x^2+y^2\leq 4} \left(-3x^2-3y^2\right)dA = -3\int_0^{2\pi}\int_0^2 r^2 r dr d\theta = -6\pi\int_0^2 r^3 dr$$

$$= (-6\pi)\left(\frac{2^4}{4}\right) = -24\pi.$$

Example 7 Evaluate : $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}=\langle y-\cos y, x\sin y\rangle$

and $C:(x-3)^2+(y+4)^2=4$

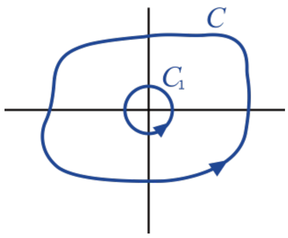
Solution : (**註：*C* 是順時鐘**) $\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r}=-\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

$$=-\iint_D(\sin y-1-\sin y)\,dA=4\pi.$$

Example 8 Let $\mathbf{F}(x,y)=\left\langle \frac{2xy}{(x^2+y^2)^2}, \frac{y^2-x^2}{(x^2+y^2)^2}\right\rangle$. Find $\int_C \mathbf{F}\cdot d\mathbf{r}=?$, where

C is any positively oriented simple curve that encloses the origin.

$C_1: x^2+y^2=a^2$.



Solution : (i) $P=\frac{2xy}{(x^2+y^2)^2},\ Q=\frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ (skip).

(ii) By **Green's Theorem**

$$0=\int_{C\cup(-C_1)}\mathbf{F}\cdot d\mathbf{r} \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}.$$

(iii) $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}$

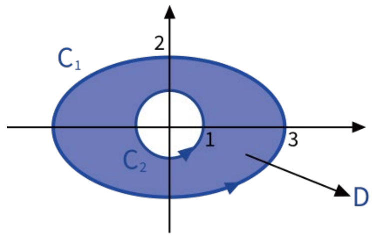
$$=\int_0^{2\pi}\left(\frac{2a^2\cos\theta\sin\theta}{a^4}a(-\sin\theta)+\frac{a^2(\sin^2\theta-\cos^2\theta)}{a^4}a(\cos\theta)\right)d\theta$$

$$=\int_0^{2\pi}\frac{-\sin^2\theta\cos\theta-\cos^3\theta}{a}d\theta=\int_0^{2\pi}\frac{-\cos\theta}{a}d\theta=0.$$

Example 9 $A=\int_C 2xydx+(x^2+2x)\,dy=?$

$C_1:\dfrac{x^2}{9}+\dfrac{y^2}{4}=1 \qquad C_2:x^2+y^2=1$

$C=C_1\cup(-C_2)$



Solution : $A=\iint_D(2x+2-2x)\,dy=2\text{Area}(D)=2(6\pi-\pi)=10\pi$

Green's Theorem can be extended to regions with holes.

Example 10 Find the positively oriented simple closed curve C for which the value of the line integral

$\int_C (y^3-y)\,dx-2x^3dy\;(=B)$

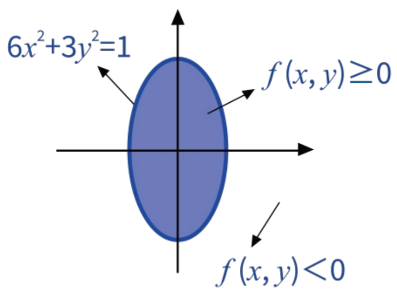
is a maximum.

Solution: By the **Green's Theorem**, we have that

$$B=\iint_D\underbrace{\left(-6x^2-3y^2+1\right)}_{f(x,y)}dA$$

$\Rightarrow C:6x^2+3y^2=1$ (逆時鐘)

會讓 B 有最大值.



以下要提 Green's Theorem 和 Divergence Theorem 及 Stoke's Theorem 的(數學)關係

(I) 和 **Stoke's Theorem** 的關係：**Stoke's Theorem** $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (curl\ \mathbf{F})\cdot \mathbf{N}dS.$

將二維的 **F** 放入三維空間：得

$$\mathbf{F}(x,y,z) = \langle P, Q, 0\rangle \Rightarrow \text{curl } \mathbf{F} = \left\langle -\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle.$$

$$\Rightarrow \text{curl } \mathbf{F}\cdot\langle 0,0,1\rangle := \text{curl } \mathbf{F}\cdot\mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Green's Theorem

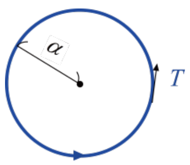
$$\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (curl\ \mathbf{F})\cdot \mathbf{k}dA.$$

$\Rightarrow$(1) **Green's Theorem** 是 **Stoke's Theorem** 放在二維平面的情境下的特殊情形，也即若 S 可放入三維空間中的 xy 平面，且 **F** 在 z 方向沒有分量，則相對應的 **Stoke's Theorem** 變成 **Green's Theorem** 的型式.

(2)二維的一個函數的某種微分型式的surface integral = 原函數一維的線積分作用在此表面的邊界 C.

物理意義：

(3) $\int_{C_\alpha} \mathbf{F}\cdot d\mathbf{r} = \int_{C_\alpha} \mathbf{F}\cdot \mathbf{T}dS =$ circulation of $\mathbf{F}$ around C_α .



C_α:半徑為 α 的圓(α 很小)

(4) $\iint_{S_\alpha} (\text{curl } \mathbf{F})\cdot \mathbf{N}dS \approx (\text{curl } \mathbf{F})\cdot\mathbf{N}(\pi\alpha^2)$

(5) 由(3), (4) $\Rightarrow (\text{curl } \mathbf{F})\cdot\mathbf{N} = \lim_{\alpha\to 0} \dfrac{\text{circulation of } \mathbf{F} \text{ around } C_\alpha}{\text{area of disk } S_\alpha}.$

= rotation of **F** about **N** .

= the measure of the rotation effect of the fluid about **N** .

(II) **和 Divergence Theorem 的關係**

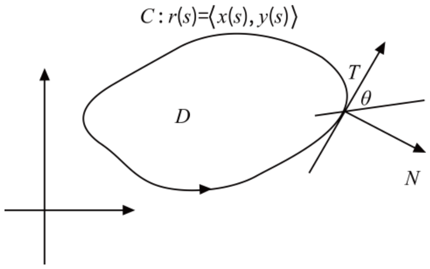
數學關係：

Divergence Theorem : $\iint_S \mathbf{F}\cdot\mathbf{N}dS=\iint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_Q \operatorname{div}\mathbf{F}\,dV.$

(一個函數的某種微分型式的三重積分作用在$Q\subset R^3$的積分
等於原函數用二重積分作用在Q的邊界S)

以上這個定理的‘特殊情形’ (二維型式)可由 **Green's Theorem** 導出

$\int_C \mathbf{F}\cdot\mathbf{N}dS=\iint_D \operatorname{div}\mathbf{F}dA$ (**Divergence Theorem** 的二維型式).



(i) 利用弧長將C參數化得$\mathbf{r}(s)=\langle x(s), y(s)\rangle$，$s$代表弧長

$$\Rightarrow \frac{d\mathbf{r}(s)}{ds}=\langle x'(s), y'(s)\rangle=\mathbf{T}=\text{unit tangent (why?)}.$$

$$\left(\text{why? } \mathbf{r}(t)=\langle x(t), y(t)\rangle \Rightarrow \frac{d\mathbf{r}(t)}{dt}=\langle x'(t), y'(t)\rangle\right.$$

$$\left.\Rightarrow T=\frac{\frac{d\mathbf{r}(t)}{dt}}{\left\|\frac{d\mathbf{r}(t)}{dt}\right\|}=\frac{\frac{d\mathbf{r}(t)}{dt}}{\|\mathbf{r}(t)\|}=\frac{\frac{d\mathbf{r}(t)}{dt}}{\frac{ds}{dt}}=\frac{d\mathbf{r}}{ds}\right).$$

(ii) $\mathbf{N}=\langle y'(s), -x'(s)\rangle$. (why?)

(iii) $\int_C \mathbf{F}\cdot\mathbf{N}dS=\int Pdy-Qdx=\int -Qdx+\int Pdy \underset{\uparrow}{=} \iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dA$

$=\iint_D \operatorname{div}\mathbf{F}\,dA.$ **Green's Theorem**

物理意義：

(iv) **F** 代表 ρ**V**, where ρ= constant density of a fluid, **v** = its velocity.

(v) $(\mathbf{F}\cdot\mathbf{N})\Delta S =$ 每單位時間流體流出 ΔS 的流量(flux).

(vi) $\int_S \mathbf{F}\cdot\mathbf{N}dS =$ the net flux of **F** across S.

(vii) 取 $S = S_\alpha$ (半徑為 α 的球表面，球心為 (x_0, y_0, z_0))

$Q = Q_\alpha$ (半徑為 α 的球體)，α 很小.

(viii)右式 $= \iiint_{Q_\alpha} \operatorname{div}\mathbf{F}\, dV \approx \operatorname{div}\mathbf{F}\, V(Q_\alpha)$.

(ix) $\operatorname{div}\mathbf{F}(x_0, y_0, z_0) \approx \dfrac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{V(Q_\alpha)}$.

(x) $\alpha \to 0 \Rightarrow \operatorname{div}\mathbf{F}(x_0, y_0, z_0) =$ flux per unit volume at (x_0, y_0, z_0).

(xi) • $\operatorname{div}\mathbf{F}(x_0, y_0, z_0) > 0$: the net flow is outward near (x_0, y_0, z_0)
(x_0, y_0, z_0) is called a source.
• $\operatorname{div}\mathbf{F}(x_0, y_0, z_0) < 0$: the net flow is inward near (x_0, y_0, z_0)
(x_0, y_0, z_0) is called a sink.
• $\operatorname{div}\mathbf{F} = 0$ for all points. Then field is called divergence free or incompressible.

(xii) Divergence Theorem：一個三維區域 Q 的進出相抵的總流量等於其二維邊界 ∂Q 進出相抵的總流量.