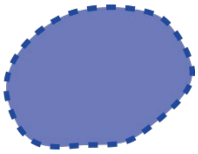
§16.3 The Fundamental Theorem for Line Integrals

＊名詞介紹：

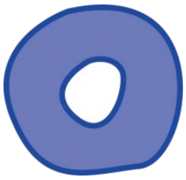
(i) smooth curves; piecewise smooth curves (= paths).

(ii) closed curves : initial point = terminal point.

(iii) open region :



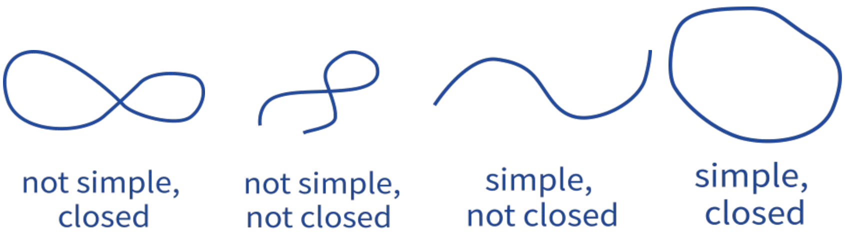
(iv) connected :



(yes)

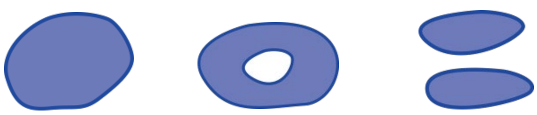
(no)

(v) simple curve : a curve that does not intersect itself anywhere between its endpoints.



(no) (yes)

(vi) Simply connected region : No hole + No separate pieces.



(yes) (no) (no)

(vii) $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path :

$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot \mathbf{dr}$ for any C_1, C_2 having the same initial points and the same terminal points.

Line integral 的基本定理

Recall：

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{(微積分基本定理, FTC)}$$

Theorem (16.3.2)(梯度定理) Let $\mathbf{F} = \langle P, Q \rangle$.

已知： (i) $\mathbf{F} = \nabla f$ (保守場).

(ii) ∇f is continuous on C.

(iii) C : (piecewise) smooth curve $\mathbf{r}(t)$, $a \le t \le b$.

結論： $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big)$.

Proof : $n = 3$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \nabla f\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\,dt \\ &= \int_a^b \left[f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \right] dt \\ &= \int_a^b \frac{d}{dt}\Big(f\big(\mathbf{r}(t)\big)\Big)\,dt \\ &\overset{\text{FTC}}{=} f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big).\end{aligned}$$

$n = 2$: Similar.

Example 1: $\mathbf{F}(x, y) = \left\langle \frac{1}{2}xy, \frac{1}{4}x^2 \right\rangle$ $C_1 : y = x$; $C_2 : x = y^2$; $C_3 : y = x^3$;

起點：$(0,0)$，終點：$(1,1)$ Compute $\int_{C_i} \mathbf{F} \cdot d\mathbf{r}$, $i = 1, 2, 3$.

Solution : $\mathbf{F} = \nabla f$, where $f(x, y) = \frac{x^2}{4} y$.

$$\Rightarrow \int_{C_i} \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(0,0) = \frac{1}{4} \quad \text{for } i = 1, 2, 3.$$

Theorem (16.3.3) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D $\Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem (16.3.4) 已知： (i) $\mathbf{F}: D \subset R^n \to R^n,\ n=2,3,\ D$: open and connected.

(ii) $\mathbf{F}$: continuous on D.

結論： $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D.

$$\Updownarrow$$

$\mathbf{F}$ is conservative.

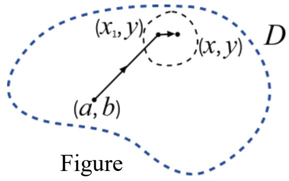
Proof: $(\Leftarrow)$ Trivial.

$(\Rightarrow)$ Constructive proof: 實際找一個 potential function f s.t. $\nabla f = \mathbf{F}$.

(i) $n=2$ $(n=3$ can be similarly proved$)$:

Let $f(x,y)=\int_C \mathbf{F}\cdot d\mathbf{r}$,

where C is a path in D with a fixed initial point (a,b) and terminal point (x,y).



Figure

(ii) $f(x,y)$ is well-defined. (What? and Why?)

(iii) Find a particular C:

Since D is open, $\exists (x_1,y)\in D$ with $x_1<x$. Let

$C=$ a path $C_1 \in D$ from (a,b) to (x_1,y) followed by the horizontal line segment C_2 from (x_1,y) to (x,y) (See above Figure).

(iv) Claim $\nabla f = \mathbf{F}$. Let $\mathbf{F}=\langle M,N\rangle$.

$$f(x,y)=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$$

$$=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2} Mdx+Ndy \qquad x=t$$

$$=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{x_1}^{x} M(t,y)\,dt. \qquad \text{Here } dy=0\ (C_2:\text{水平線段})$$

$$\frac{\partial f}{\partial x}=0+M(x,y).\ (\int_C \mathbf{F}\cdot d\mathbf{r}\ \text{是和}x\text{無關的相對常數})$$

Similar Construction (how?), we have

$$\frac{\partial f}{\partial y}=N(x,y).$$

有無檢查 F 是否保守場的方法：

$\boxed{n=2}$

已知： (i) D (定義域) : open simply-connected region.

(ii) $\mathbf{F}=\langle M,N\rangle$: M,N: continuous first-order partial derivatives.

結論： $\mathbf{F}$ is conservative on $D \Leftrightarrow \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

$(\Rightarrow)$ Clairaut's Theorem.

$(\Leftarrow)$ Green's Theorem (§16.4).

$\boxed{n=3}$ $\mathbf{F}=\langle M,N,P\rangle$

同 $n=2$ 類似的已知.

結論： $\mathbf{F}$ is conservative on $D \Leftrightarrow$ Curl $\mathbf{F}=\mathbf{0}$ for all points in D.

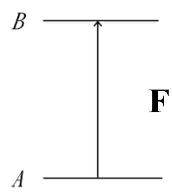
Theorem (Conservation of Energy) : In a conservative vector field,
the sum of its potential energy and its kinetic energy remains a constant at any point.
This is the reason why the vector field is called conservative.

Proof :

(i) Let $\mathbf{F}=\nabla f$

$P(A)=$ 位能 at A 點 $=-f(A)$ (why?)

$K(A)=$ 動能 at A 點 $=\dfrac{1}{2}m\|\mathbf{r}'(A)\|^2$.



$\mathbf{F}=$ Gravitaional field

Wanted : $P(A)+K(A)=P(B)+K(B)$.

$$\int_A^B \mathbf{F}\cdot d\mathbf{r}=f(B)-f(A)=P(A)-P(B). \qquad \mathbf{F}=m\underset{\substack{\uparrow\\ \text{加速度}}}{\boldsymbol{a}}=m\mathbf{r}''(t.)$$

$$\|$$

$$\int_A^B m\mathbf{r}''\cdot\mathbf{r}'dt=\frac{1}{2}m\int_A^B\frac{d}{dt}\|\mathbf{r}'(t)\|^2=\frac{1}{2}m\left(\|\mathbf{r}'(B)\|^2-\|\mathbf{r}'(A)\|^2\right)=K(B)-K(A).$$

$$\Rightarrow P(A)+K(A)=P(B)+K(B).$$