§16.2 Line (Curve) Integrals

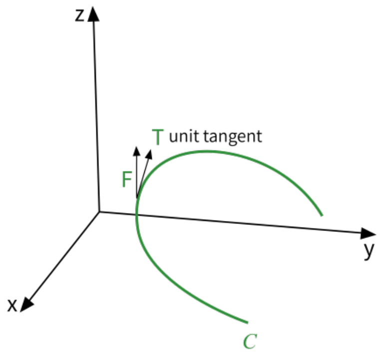
名詞:

(i) $C = \mathbf{r}(t),\ a \le t \le b$, is a smooth curve：$\mathbf{r}'(t)$ is a continuous on $[a,b]$ and $\mathbf{r}'(t) \ne 0$ for $a < t < b$.

(ii) C is a piecewise smooth curve：$C = C_1 \cup C_2 \cup \ldots \cup C_n,\ C_i, i = 1, 2, \ldots, n$, are smooth curves.

<u>4 類型的 line integrals</u>

(i) $\mathbf{F}$: vector fields (a Force). Work done by $\mathbf{F}$ along C.



$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int M\ dx + N\ dy, \text{ where } \mathbf{F} = \langle M, N\rangle \text{ and } d\mathbf{r} = \langle dx, dy\rangle$$

= the line integral of $\mathbf{F}$ along C.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt$$

$$= \int_a^b \mathbf{F}(r(t))\cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|\,dt$$

$$= \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

(ii) $\int_C f(x,y)\,ds = \int_C f(x,y)\sqrt{(dx)^2+(dy)^2} = \int_a^b f(x(t),y(t))\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt.$

(line integral of f along C with respect to arc length)

(iii) $\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt.$

(iv) $\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt.$

Facts：

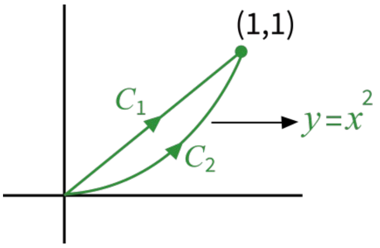
(i) $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds.$

(ii) $\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx.$

(iii) $\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy.$

(iv) $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}.$

Example 1：



Let $\mathbf{F}(x,y)=\langle x,0\rangle$,

$C_1:\mathbf{r}(t)=\langle t,t\rangle,\quad C_2:\mathbf{r}(t)=\langle t,t^2\rangle,\ 0\le t\le 1,$

$-C_1:\mathbf{r}(t)=\langle 1-t,1-t\rangle,\ 0\le t\le 1.$

Solution：

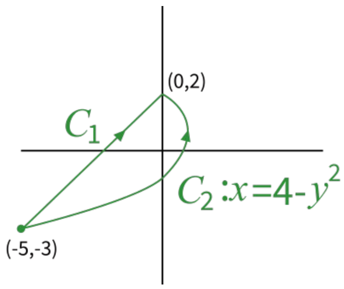
(i) $\int_{C_1} x\,ds=\int_0^1 t\sqrt{2}\,dt=\frac{\sqrt{2}}{2}\Big|_0^1=\frac{\sqrt{2}}{2}\qquad \int_{-C_1} x\,ds=\int_0^1(1-t)\sqrt{2}\,dt=-\frac{\sqrt{2}(1-t)^2}{2}\Big|_0^1=\frac{\sqrt{2}}{2}.$

(ii) $\int_{C_1} x\,dx=\int_0^1 t\,dt=\frac{1}{2}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}=\langle x,0\rangle$, $\int_{-C_1} x\,dx=\int_0^1(1-t)\,d(-t)=-\frac{1}{2}.$

(iii) $\int_{C_2} x\,ds=\int_0^1 t\sqrt{1+4t^2}\,dt=\frac{2}{3}\left(1+4t^2\right)^{\frac{3}{2}}\left(\frac{1}{8}\right)\Big|_0^1=\frac{1}{12}\left(5\sqrt{5}-1\right).$

(iv) $\int_{C_2} x\,dx=\int_0^1 t\,dt=\frac{1}{2}=\int_{C_2}\mathbf{F}\cdot d\mathbf{r}.\qquad \mathbf{F}=\langle x,0\rangle$

Example 2： $\mathbf{F}=\left\langle y^2,x\right\rangle$. Compute $\int_{C_i}\mathbf{F}\cdot d\mathbf{r}$, $i=1,2$, C_i如下圖.



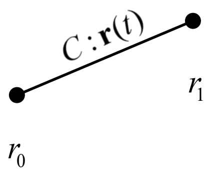
$\int_{C_1} y^2\,dx+x\,dy$

Solution： $C_1:\mathbf{r}(t)=(1-t)\left\langle -5,-3\right\rangle+t\left\langle 0,2\right\rangle=\left\langle 5t-5,5t-3\right\rangle\quad 0\le t\le 1$

$$\Rightarrow\int_{C_1} y^2\,dx+x\,dy=\int_0^1\left[(5t-3)^2\,5+(5t-5)5\right]dt=-\frac{5}{6}$$

$$\int_{C_2} y^2\,dx+x\,dy=\int_{-3}^{2}\left[y^2(-2y)+\left(4-y^2\right)\right]dy=40\frac{5}{6}.$$

技術：直線參數式



$C:\mathbf{r}(t)=(1-t)r_0+tr_1$

在 Example 1 $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, 但 Example 2 中 $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}\ne\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$

(答案：在 Example 1 中的 **F** 是 conservative, 但 Example 2 中的 **F** 不是, 在§16.3 會看得更清楚).

Example 3： Evaluate $\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$, where

C : line segments from $\overline{(0,0,0) \text{ to } (1,0,1)}^{C_1}$ and from $\overline{(1,0,1) \text{ to } (0,1,2)}^{C_2}$.

Solution 1 : $C_1 : \langle t, 0, t\rangle \qquad C_2 : (1-t)\langle 1,0,1\rangle + t\langle 0,1,2\rangle = \langle 1-t, t, 1+t\rangle$

$\mathbf{F} = \langle y+z, x+z, x+y\rangle$

$d\mathbf{r} = \langle dx, dy, dz\rangle$

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_0^1 (t+t)\,dt + \int_0^1 \big(-(1+2t)+2+1\big)\,dt$$

$$= t^2\Big|_0^1 + (2t-t^2)\Big|_0^1 = 1+1 = 2.$$

Solution 2 : (16.3 線積分的基本定理)

$f(x,y,z) = xy + xz + yz \quad \Rightarrow \nabla f = \mathbf{F}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= f(1,0,1) - f(0,0,0) + f(0,1,2) - f(1,0,1) = 2.$$

Example 4: Evaluate $\int \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle \sin x, \cos y, xz\rangle$ and

$\mathbf{r}(t) = \langle t^3, -t^2, t\rangle$, $0 \le t \le 1$.

Solution : $\int \mathbf{F}\cdot d\mathbf{r} = \int_0^1 3t^2 \sin t^3)\,dt + \int_0^1 -2t\cos(-t^2)\,dt + \int_0^1 t^4\,dt$

$$= -\cos t^3\Big|_0^1 + \sin(-t^2)\Big|_0^1 + \frac{t^5}{5}\Big|_0^1$$

$$= -\cos 1 + 1 - \sin 1 + \frac{1}{5}$$

$$= \frac{6}{5} - \cos 1 - \sin 1.$$