向量微積分(向量分析)

符號：粗體代表向量，如 $\mathbf{r}$, $\mathbf{F}$; 正常體代表純量，如 r, F

廣泛應用於：物理和工程，電磁場和流體的動態.

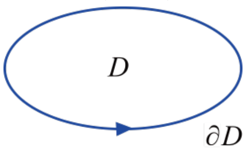
數學上學什麼？

(I) :幾個微分有關的量

1. **Gradient**(梯度): grad $(f) = \nabla f$ (向量場).
   描述在某點附近，純量場 f 增加率最大的方向.
2. **Divergence**(散度): $\operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F}$ (純量場).
   描述在某點附近，向量場 $\mathbf{F}$ 的發散或匯聚的程度.
3. **Curl**(旋度): $\operatorname{curl}(\mathbf{F}) = \nabla\times\mathbf{F}$ (向量場).
   描述在某點附近，向量場 $\mathbf{F}$ 的旋轉程度.

(II) :四個重要的定理(都是微積分基本定理的推廣，尤其是高維度的推廣)

1. **梯度定理** (一維: 曲線) : $\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$,
   $\mathbf{b}$ and $\mathbf{a}$ 是曲線 C 的終點和起點.
   一個函數的某種微分型式在某一 n 維區域的 n 重積分量(積分)=原函數在此區域邊界的 $(n-1)$ 重積分量(積分).
2. **Green's Theorem** (二維) : $\iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dA = \int_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \int_{\partial D} P\,dx + Q\,dy.$
   $\mathbf{F} = \langle P, Q\rangle$



3. **Stokes' Theorem** : $\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\cdot d\mathbf{r}.$
4. **Divergence Theorem** (三維) : $\iiint_{E\subset R^3} \operatorname{div}(\mathbf{F})\,dV = \iint_{\partial E} \mathbf{F}\cdot d\mathbf{S}.$
5. **Green's Theorem** $\subset$ **Stokes' Theorem**
   $\Rightarrow$ **Divergence Theorem** .

§16.1 Vector Fields

符號：粗體代表向量，如 **r**, **F** ；正常體代表純量，如 r, F

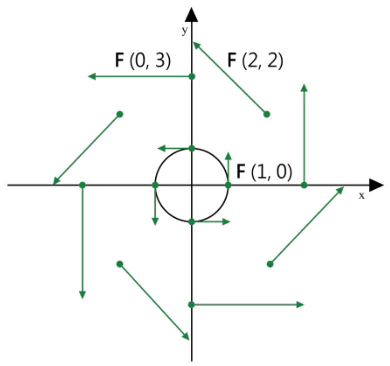
主題：

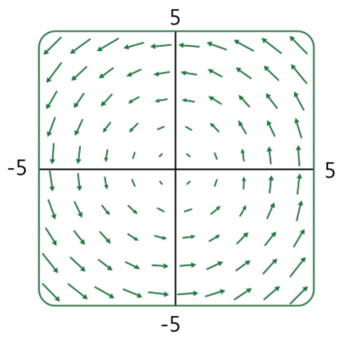
(i) Vector Field **F**：$\mathbf{F}: D \subset R^n \to R^n$, $n = 2, 3$.

(ii) Conservative Vector Field **F**：$\exists f : R^n \to R$ s.t. $\nabla f = \mathbf{F}$;
Such f is called a potential function of **F**.

(iii) Curl and Divergence of **F**.

Example 1： $\mathbf{F}(x, y) = \langle -y, x \rangle$

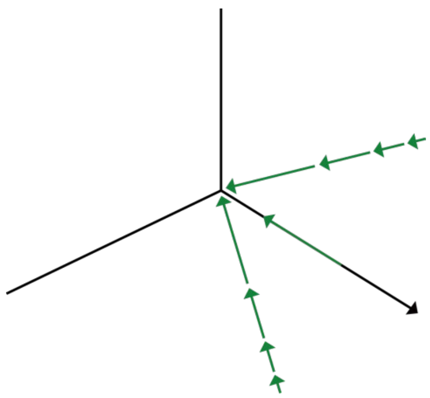




Example 2： Gravitation Force (Field) $\mathbf{F}(\mathbf{x}) = -\dfrac{GmM}{|\mathbf{x}|^3}\mathbf{x}$;

m, M are masses of two objects, $\mathbf{x} = (x, y, z)$.

$\left(\text{Gravitation} \propto \dfrac{mM}{r^2} \right)$



Example 3： Electric Force (Field) $\mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$ q, Q are electric charges (電荷).

(i) $qQ > 0 \Rightarrow$ the force is repulsive.

(ii) $qQ < 0 \Rightarrow$ the force is attraction.

Example 4：Gradient Fields ∇f : 此處 f 是個多變數的存量函數

也即 $f : R^n \to R,\ n > 1.$

Definition: a vector field **F** is called a **conservative vector field** if $\exists$ a scalar function f s.t. $\nabla f = \mathbf{F}$.

(ii) - a：Given **F**, check if **F** is conservative vector field (CVF)?
(ii) - b：If yes, how to find a potential function f?

Example 1： $\mathbf{F}(x, y) = \langle -y, x\rangle = \langle M, N\rangle$.   Is **F** a CVF?

Solution：If $\exists f$ s.t. $\mathbf{F} = \nabla f = \langle f_x, f_y\rangle$, then $f_x = M$ and $f_y = N$,

However, $\dfrac{\partial M}{\partial y} = f_{xy} = -1$ and $\dfrac{\partial M}{\partial x} = f_{yx} = 1$. Hence, $f_{xy} \neq f_{yx}$, $(\rightarrow\leftarrow)$.

Thus, **F** is not conservative .

Example 2： Is the gravitational field **F** a CVF?

$$\mathbf{F}(\mathbf{x}) = \text{Gravitational Field} = -\frac{GmM}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\langle x, y, z\rangle.$$

Solution：$f\langle x, y, z\rangle = \dfrac{GmM}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$.

Example 3： $\mathbf{F}(x, y) = \langle 2xy, x^2 - y^2\rangle = \langle M, N\rangle \overset{?}{=} \langle f_x, f_y\rangle$.

Find a potential function f of **F**.

Solution：$f(x, y) = x^2 y + g_1(y) = x^2 y - \dfrac{y^2}{2} + g_2(x)$.

$$\Rightarrow f(x, y) = x^2 y - \frac{y^2}{2} + k.$$

Theorem :(檢查**F**,二維,是否是保守場的充分必要條件)

已知： (i) Let $\mathbf{F}: D \subset R^n \rightarrow R^n$, D : open desk , $\mathbf{F} = \langle M, N\rangle$.

(ii) M, N have continuous first partial derivatives on R.

結論： $\mathbf{F} = \langle M, N\rangle$ is conservative $\underset{\text{Theorem 16.3.6}}{\overset{\text{Theorem 16.3.5}}{\rightleftarrows}} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Remark： 若已知(i),(ii)滿足，且 $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$， 則 **F** is not conservative.

How about the counterpart Theorem in space？
(三維的相對定理)
先介紹　curl **F** (**F** 的旋度)

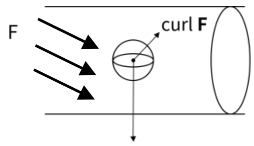
Definition：Let $\mathbf{F}(x, y, z) = \langle M, N, P\rangle$

Define：$\text{Curl}\,\mathbf{F}(x, y, z) = \nabla \times F(x, y, z) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle M, N, P\rangle$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle.$$

- Physical (Intuitive) Interpretation of the curl of **F**. (見習題§16.5-37)
  F：velocity field of a fluid flow　中放入一個球心固定的球



則　球的轉軸 = curl **F**

球的轉速 = $\frac{1}{2}\left|\text{ curl }\mathbf{F}(\text{球心})\right|$.

- If curl **F** = 0, then **F** is said to be irrotational.
- Curl **F**(x, y, z)代表向量場 **F** 在(x, y, z)附近的旋轉的方向和大小程度.

Theorem:(檢查**F**,三維,是否是保守場的充分必要條件)

已知：$F = \langle M, N, P\rangle$,　M, N, P: continuous 1st partial derivatives on an open space in space Q.

結論：**F** is conservative . $\underset{\text{Theorem 16.6.4}}{\overset{\text{Theorem 16.5.3}}{\rightleftarrows}}$ curl **F**(x, y, z) = **0**, for all (x, y, z) in Q.

Divergence (散度) of a Vector Field $\mathbf{F} = \langle M,\ N,\ P \rangle$.

- $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = M_x + N_y + P_z$.
- If $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F}$ is said to be incompressible .
- $\operatorname{div} \mathbf{F}(x, y, z)$ = the net rate of outward flux per unit volume at (x, y, z) .

  =描述在(x, y, z)點附近，向量場 $\mathbf{F}$ 的發散或匯聚的程度.
- $\operatorname{div} \mathbf{F}(x, y, z) \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \Rightarrow (x, y, z)$ → a source, / incompressible, / a sink.