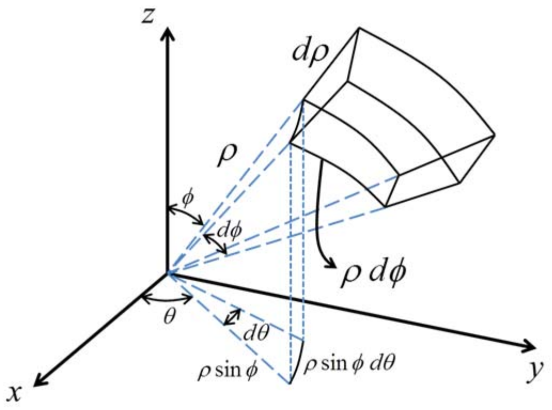
§15.9 Triple Integrals in Spherical Coordinates

∗Spherical Coordinates(S.C.)

$E =$ 球狀物體



Volume element in SC
$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Let $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$

$$\iiint_E f(x,y,z)dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho,\theta,\phi)\rho^2 \sin\phi \, d\rho d\theta d\phi.$$

Example 1：

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV$$

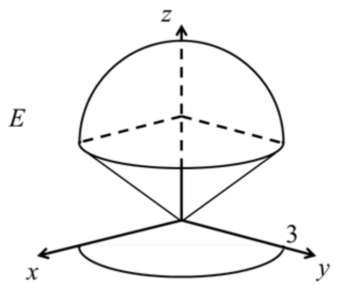
Solution：

$$\int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi d\rho d\theta d\phi = \frac{4\pi}{3}(e-1).$$

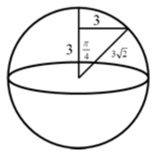
Example 2：

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}\left(x^2+y^2+z^2\right)dzdxdy.$$

Solution：



$$\sqrt{x^2+y^2}=\sqrt{18-x^2-y^2}$$
$$\Rightarrow x^2+y^2=9$$



$$=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{3\times2^\wedge\left(\frac{1}{2}\right)}\rho^2\rho^2\sin\phi d\rho d\theta d\phi$$

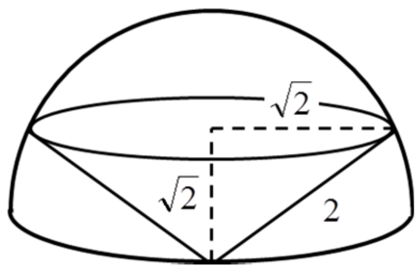
$$=486\pi\left(\frac{\sqrt{2}-1}{5}\right).$$

Example 3：

Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy – plane and below the cone $z=\sqrt{x^2+y^2}$.

Solution：

$$\begin{cases}x^2+y^2+z^2=4\\ \sqrt{x^2+y^2}=z\end{cases}\Rightarrow x^2+y^2=2.$$



$$V=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 1\times\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{8\sqrt{2}}{3}\pi.$$

Example 4：

Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.

Solution：

$$V(E)=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\phi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi}d\phi=\frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\sin\phi\cos^3\phi d\phi=\frac{\pi}{8}.$$