§15.8 Triple Integrals in Cylindrical Coordinates

∗Cylindrical Coordinates (C.C.)

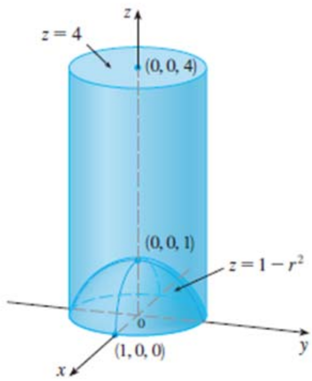
$E = D \times [u_1(x,y), u_2(x,y)]$ (圓柱狀適用C.C.)

D : 用polar coordinate來表達

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_2(\theta)}^{h_1(\theta)} \int_{u_1}^{u_2} rf(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Example 1：

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



Example 2：

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$

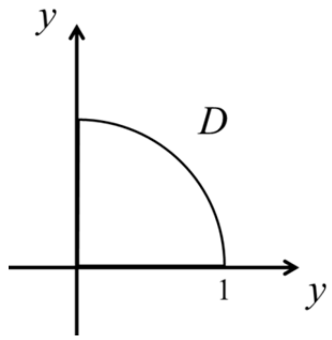
Solution：

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx &= \iiint_E x^2+y^2\,dV \\
&= \int_0^{2\pi}\int_0^2\int_r^2 r^2 r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)\,dr \\
&= 2\pi\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 = \frac{16}{5}\pi.
\end{aligned}$$

Example 3：

$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$$

Solution：



$$= \iint_D \frac{xy}{2}\left[x^2+y^2-\left(x^2+y^2\right)^2\right]dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \frac{\sin 2\theta}{4}\left(r^2-r^4\right)r^3\,dr\,d\theta$$

$$= \frac{1}{96}.$$