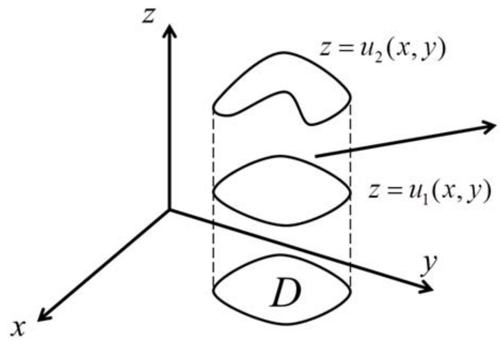
§15.7 Triple Integrals

＊屋頂：$u = f(x, y, z)$

地基：在三維空間的物體 E.



$$E = D \times [u_1(x,y), u_2(x,y)]$$
$$= \{(x,y,z) : x, y \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

＊Triple Integral：

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right]dA \qquad (1)$$

＊註記：

Fubini Theorem：屋頂夠好(沒三維破洞，或只有有限的二維或一維的裂縫)，則 Triple Integral 可寫成遞回式(iterated)的三個一維積分。

Example 1：

$\iiint_E zdV$, where E is the solid bounded by the four planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

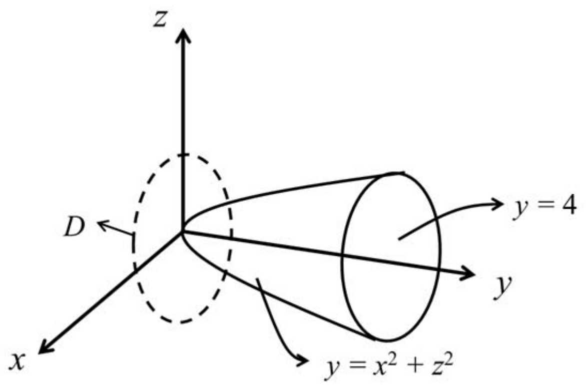
Solution：

$$\iiint_E zdV$$
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} zdzdydx$$
$$= \frac{1}{24}$$
$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} zdxdydz$$

Example 2：

$$\iiint_E \sqrt{x^2+z^2}\,dV.\ E \text{ is bounded by } \begin{cases} y = x^2 + z^2 \\ y = 4 \end{cases}.$$

Solution：

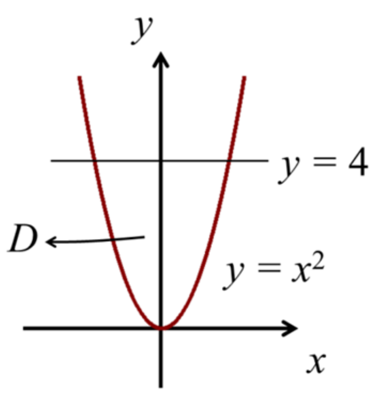


$$E = D \times \left[x^2 + z^2, 4\right]$$
$$D = x^2 + z^2 \le 4.$$

$$\Rightarrow \iiint_E \sqrt{x^2+z^2}\,dV$$
$$= \iint_D \left(\int_{x^2+y^2}^{4} \sqrt{x^2+z^2}\,dy \right) dA$$
$$= \iint_{x^2+z^2\le 4} \left(\sqrt{x^2+z^2}\right)\left(4-x^2-z^2\right) dA$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx$$
$$= \int_0^{2\pi}\int_0^{2} r^2\left(4-r^2\right) dr\,d\theta$$
$$= \frac{128\pi}{15}.$$

Alternate solution：

$$E = D' \times \left[-\sqrt{y-x^2}, \sqrt{y-x^2}\ \right]$$



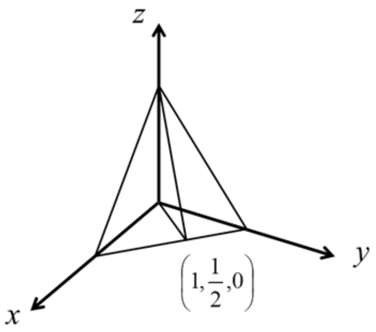
$$\Rightarrow \iint_{D'} \left(\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz \right) dA$$

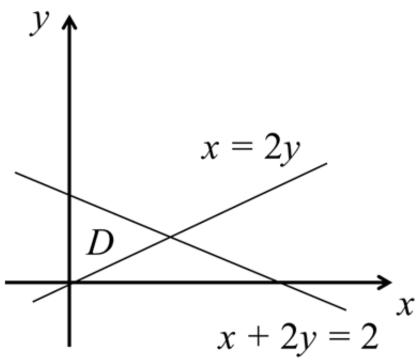
$$= \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz dy dx \text{ (較難算)}$$

Example 3：

Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Solution：





$$V = \iint_{D} \int_{0}^{2-x-2y} 1\, dz dA$$

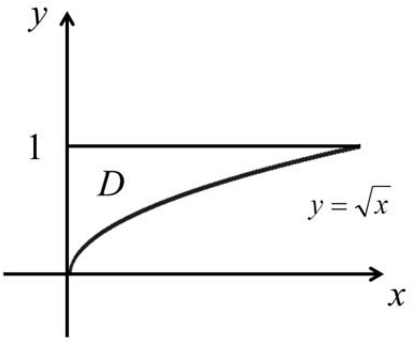
$$= \int_{0}^{1} \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_{0}^{2-x-2y} 1\, dz dy dx$$

$$= \frac{1}{3}.$$

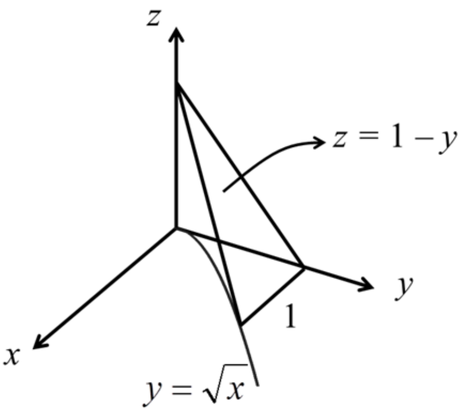
Example 4：

Rewrite $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ as an equivalent iterated integral in the five other orders.

Solution：



$$E = D\times[0,1-y] = \{(x,y,z):(x,y)\in D \text{ and } 0\le z\le 1-y\}$$



$$\Rightarrow \iint\limits_D \int_0^{1-y} f\,dz\,dA$$

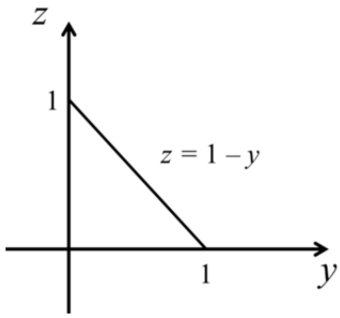
$$= \int_0^1\int_0^{y^2}\int_0^{1-y} f\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-z}\int_0^{y^2} f\,dx\,dy\,dz$$

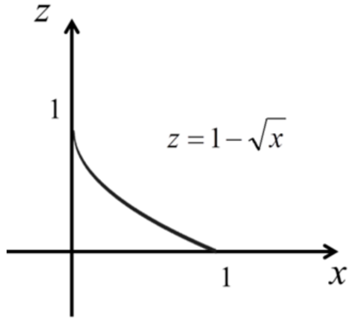
$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz$$



$$\begin{cases} y=\sqrt{x} \\ z=1-y \end{cases} \Rightarrow z=1-\sqrt{x}$$



Example 5：

Which of the following iterate integrals are equal to the integral?

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^x f(x,y,z)\,dzdydx.$$

(A) $\int_0^1 \int_0^z \int_0^{\sqrt{1-x^2}} f(x,y,z)\,dydzdx$

(B) $\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} f(x,y,z)\,dydzdx$

(C) $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_z^{\sqrt{1-z^2}} f(x,y,z)\,dxdydz$

(D) $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_z^{\sqrt{1-y^2}} f(x,y,z)\,dxdydz$

Solution：

(B),(D).