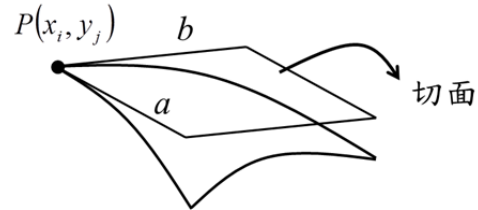# §15.6 Surface Area

＊單變數函數

　弧長→用線段逼近

＊多變數函數

　Surface Area→用切面面積逼近

過 P_{ij} 的切面來逼近
過 P_{ij} 的曲面.

（ $P_{ij} = P(x_i, y_j)$ ）

$a = \Delta x i + 0 j + f_x \Delta x k$

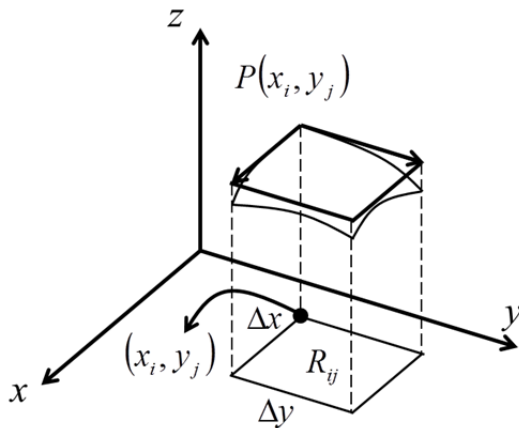$b = 0 i + \Delta y j + f_y \Delta y k$

曲面面積 $= \Delta S \approx$ 切面面積

$\Rightarrow$ 切面面積 $= |a \times b|$

$= \sqrt{1 + f_x^2 + f_y^2} \, (\Delta x \Delta y)$

$= \sqrt{1 + f_x^2 + f_y^2} \, (\Delta A)$

$\Rightarrow$ 作無窮切割後

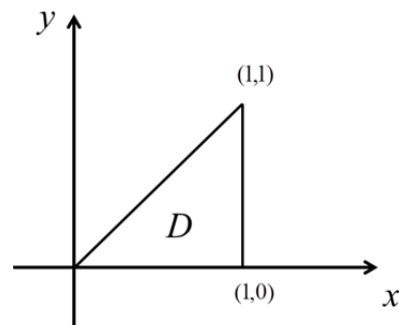$dS = \sqrt{1 + f_x^2 + f_y^2} \, dA$

＊公式：地基為 D，屋頂為 $z = f(x, y)$.

$\Rightarrow$ 其屋頂面積 $= S = \iint\limits_{D} dS = \iint\limits_{D} \sqrt{1 + f_x^2 + f_y^2} \, dA.$

Example 1：

$z = x^2 + 2y$

Solution：

$S = \int_0^1 \int_x^1 \sqrt{1 + (2x)^2 + 2^2} \, dy \, dx$

$= \dfrac{1}{12} \left(27 - 5\sqrt{5} \right)$

Example 2：

Find surface area of $z = x^2 + y^2$ under $z = 9$.

Solution：

$$S = \iint\limits_{x^2 + y^2 \leq 9} \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

Example 3：

Find the surface area of $z = y^2 - x^2$ between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution：

$$S = \iint\limits_{1 \leq x^2 + y^2 \leq 4} \sqrt{1 + 4y^2 + 4x^2}\, dA = 極座標轉換$$

Example 4：

Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$.

Solution：

$$地基 \begin{cases} x^2 + z^2 = y \\ y = 25 \end{cases} \Rightarrow x^2 + z^2 = 5^2 \ (y = 0)$$

屋頂　$y = x^2 + z^2$

$$S = \iint\limits_{x^2 + y^2 \leq 5^2} \sqrt{1 + 4x^2 + 4z^2}\, dA$$

$$= \int_0^{2\pi} \int_0^5 r\sqrt{1 + 4r^2}\, dr\, d\theta$$

$$= \frac{\pi}{6}\left(101\sqrt{101} - 1\right)$$