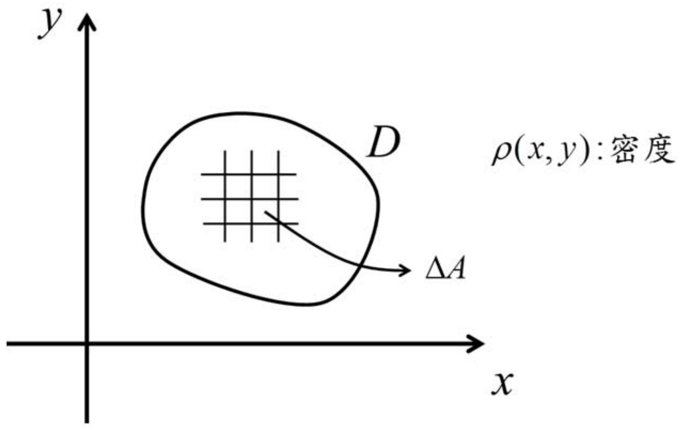
§15.5 Applications of Double Integrals

I. Density and Mass



ΔA_{ij}：細切割的小面積，在此小區域中 $\rho(x,y)$ 差別不太大，可取其中一點 (x_{ij}^*, y_{ij}^*) 作為密度。

$$\text{此小區域的重量} = \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

$$\text{整個切割區域的重量} = \sum_{i,j} \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

$$\xrightarrow{\text{無窮切割}, i,j\to\infty} \text{總重量} = \iint_D \rho(x,y)\,dA.$$

＊結論：

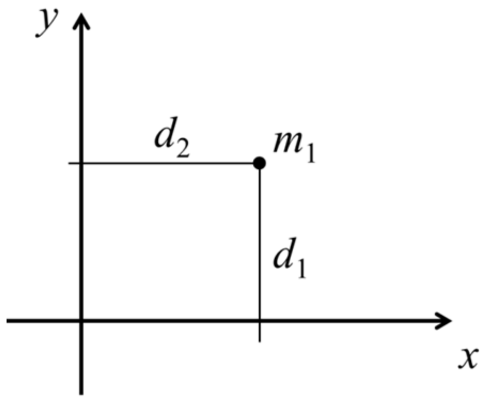
(i). 若 $\rho(x,y)$ 代表 D 區域的 density

$\Rightarrow$ total mass $m = \iint_D \rho(x,y)\,dA.$

(ii). 若 $\rho(x,y)$ 代表 electric charge 在 D 區域的強度(charge density)

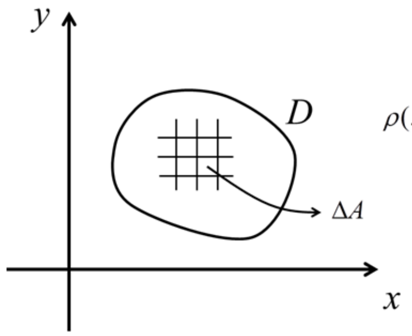
$\Rightarrow$ total charge $Q = \iint_D \rho(x,y)\,dA.$

II. Moments and Centers of Mass



M_x (with respect to a particle) $= m_1 d_1$

M_y (with respect to a particle) $= m_1 d_2$



$\rho(x, y)$：密度

$$M_x = \iint_D y\rho(x, y)\,dA$$

$$M_y = \iint_D x\rho(x, y)\,dA$$

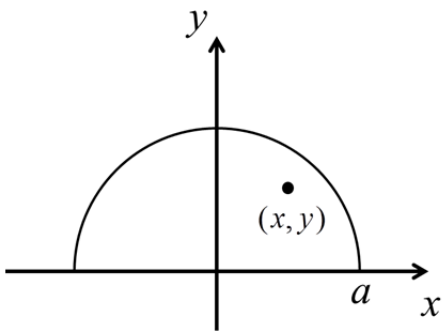
$\Rightarrow$ The center $(\bar{x}, \bar{y})$ of mass of a lamina on D with density $\rho(x, y)$ is

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\,dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\,dA$$

Example 1：

The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of the lamina.



Solution：

$\rho(x,y)$對稱 y 軸 $\Rightarrow$ center 在 y 軸上

$\Rightarrow$ 只要算 $\bar{y}$ 即可

$$\Rightarrow \bar{y} = \frac{\iint_D k\sqrt{x^2+y^2}\,y\,dA}{\iint_D k\sqrt{x^2+y^2}\,dA} = \frac{\iint_D \sqrt{x^2+y^2}\,y\,dA}{\iint_D \sqrt{x^2+y^2}\,dA}$$

分母：$\int_0^{\pi}\int_0^{a} r^2\,dr\,d\theta = \frac{\pi a^3}{3}$ 　　分子：$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\,dr\,d\theta = \frac{a^4}{2}$

$$\Rightarrow \bar{y} = \frac{\frac{a^4}{2}}{\frac{\pi a^3}{3}} = \frac{3a}{2\pi}.$$

III. Probability：

＊A pair of continuous random variables X & Y.(e.g. X：身高 , Y：體重.)

Let $f(x, y)$ be the joint density function. Then

(i). $f(x,y) \ge 0.$

(ii). $\iint_{R^2} f(x,y)dA = 1.$

(iii). $P\big((X,Y) \in D\big) = \iint_{D} f(x,y)dA.$

＊Expected values：

(i). X：random variable with probability density function f.

$\Rightarrow$ mean $\mu = \int_{-\infty}^{\infty} xf(x)fx.$

(ii). X and Y：random variables with joint density function $f(x, y)$, then

1. $X-$mean (expected values of X) $= \iint_{R^2} xf(x,y)dA$

2. $Y-$mean (expected values of Y) $= \iint_{R^2} yf(x,y)dA$