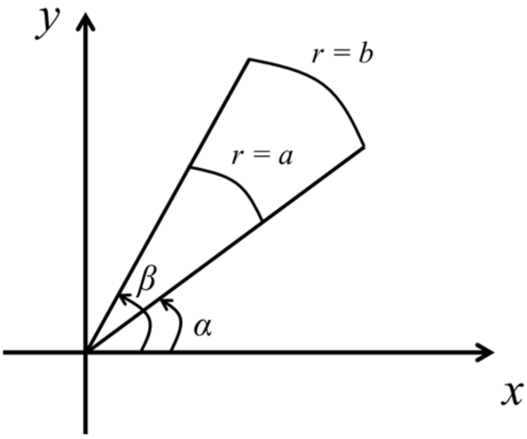
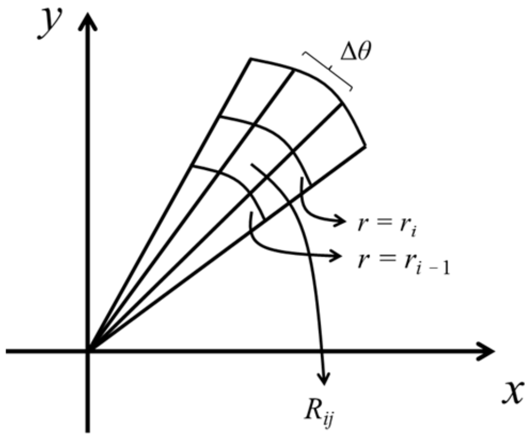
§15.4 Double Integrals in Polar Coordinates

＊極座標積分





$R=\{(r,\theta):a\le r\le b,\alpha\le\theta\le\beta\}$,將角度和半徑切很細時

$A_{ij}=\text{Area of }R_{ij}=\frac{1}{2}\left(r_i^2-r_{i-1}^2\right)\Delta\theta=\frac{1}{2}(r_i+r_{i-1})\Delta r\Delta\theta$

當等分成"無窮"細時

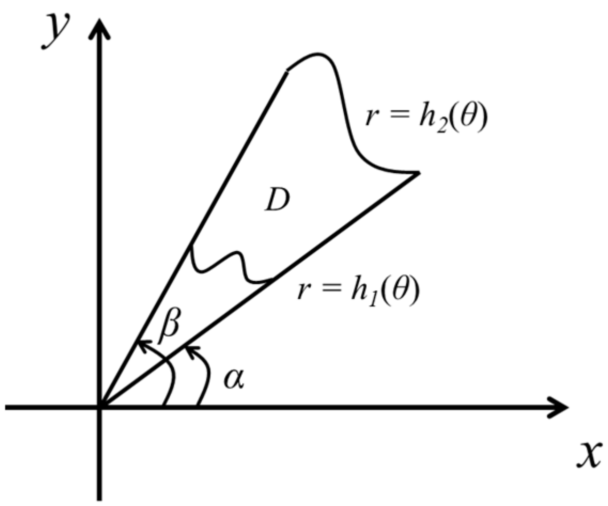
$A_{ij}\to dA=rdrd\theta.$

Theorem：

(i). $\iint\limits_R f(x,y)dA=\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)\,rdrd\theta$

(ii). $D=\{(r,\theta):\alpha\le\theta\le\beta,h_1(\theta)\le r\le h_2(\theta)\}$

$\Rightarrow\iint\limits_D f(x,y)dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,r\sin\theta)rdrd\theta.$



Example 1：

$$\iint_D (3x+4y^2)dA.\ D: 1 \le x^2+y^2 \le 4,\ y \ge 0.$$

Solution：

$$\iint_D \left(3x+4y^2\right)dA$$

$$= \int_0^{\pi}\int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \frac{15\pi}{2}$$

$$= \int_{-2}^{2}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \left(3x+4y^2\right)dy\,dx.$$

Example 2：

Find the volume of $\begin{cases} z = 1-x^2-y^2 \\ z = 0 \end{cases}$

Solution：

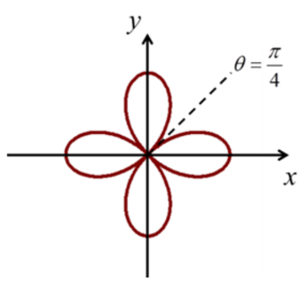
$$\iint_D \left(1-x^2-y^2\right)dA \quad D: x^2+y^2 \le 1.$$

$$= \int_0^{2\pi}\int_0^1 \left(1-r^2\right) r\,dr\,d\theta = \frac{\pi}{2}.$$

Example 3：

Find the area of $r=\cos 2\theta$ (Area = Volume with height 1)

Solution：



$$\text{Area} = 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\cos 2\theta} (1)\,r\,dr\,d\theta$$

$$= 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= \frac{\pi}{2}$$

Example 4：

Find the volume of the solid that lies $\begin{cases} \text{under } z = x^2+y^2 \\ \text{inside } x^2+y^2 = 2x. \\ \text{above } z=0 \end{cases}$

Solution：

$$V=\iint_D\left(x^2+y^2\right)dA$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}r^2 r\,dr\,d\theta$$
$$=\frac{3\pi}{2}$$

地基

$D:(x-1)^2+y^2=1$

$D:r=2\cos\theta$

屋頂

$z=x^2+y^2$

$z=r^2$

Example 5：

Find the volume of $\begin{cases}x^2+y^2=4\\ \dfrac{x^2}{16}+\dfrac{y^2}{16}+\dfrac{z^2}{64}=1\end{cases}$

Solution：

$$V=2\int_0^{2\pi}\int_0^2 2\sqrt{16-r^2}\,r\,dr\,d\theta$$
$$=\frac{8\pi}{3}\left(64-24\sqrt{3}\right)$$

地基

$D:x^2+y^2=4$

$D:r=2$

屋頂

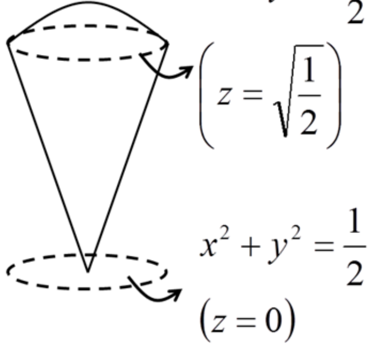
$$z=\sqrt{64-4x^2-4y^2}$$
$$=\sqrt{64-4r^2}$$
$$=2\sqrt{16-r^2}$$

Example 6：

Find the volume of the solid that lies $\begin{cases}\text{below } x^2+y^2+z^2=1\\ \text{above } z=\sqrt{x^2+y^2}\end{cases}$

Solution：

$$\begin{cases}x^2+y^2+z^2=1\\ z=\sqrt{x^2+y^2}\end{cases}$$
$$\Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$$
$$\Rightarrow x^2+y^2=\frac{1}{2}.$$



地基

$$V=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(\sqrt{1-r^2}-r\right)rdrd\theta$$

$$=\frac{\pi}{3}\left(2-\sqrt{2}\right)$$

$$D: x^2+y^2=\frac{1}{2}$$

$$r=\frac{\sqrt{2}}{2}$$

屋頂

$$z=\sqrt{1-x^2-y^2}=\sqrt{1-r^2}$$

$$z=\sqrt{x^2+y^2}=r$$

Example 7：

Evaluate the iterated integral by converting to polar coordinates

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dydx.$$

Solution：

$$y=\sqrt{2x-x^2}\Rightarrow x^2+y^2=2x\Rightarrow r=2\cos\theta$$

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dydx$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}r^2drd\theta$$

$$=\frac{16}{9}.$$

Example 8：

Use polar coordinates to combine the sum

$$\int_{\frac{1}{\sqrt{2}}}^1\int_{\sqrt{1-x^2}}^x xydydx+\int_1^{\sqrt{2}}\int_0^x xydydx+\int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} xydydx$$

into one double integral. Then evaluate the double integral.

Solution：

$$\frac{15}{16}.$$