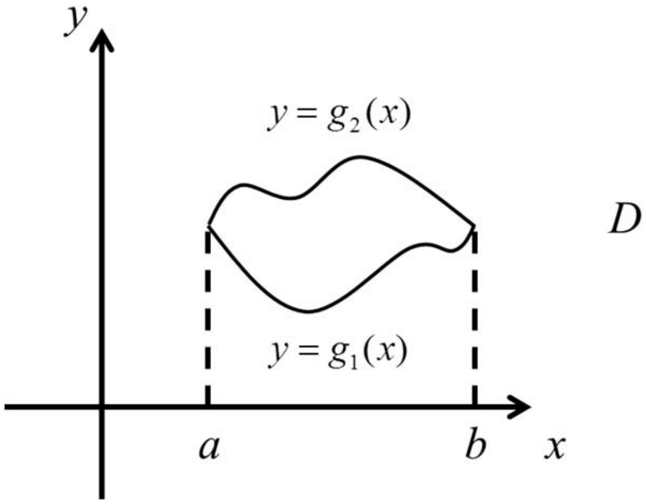
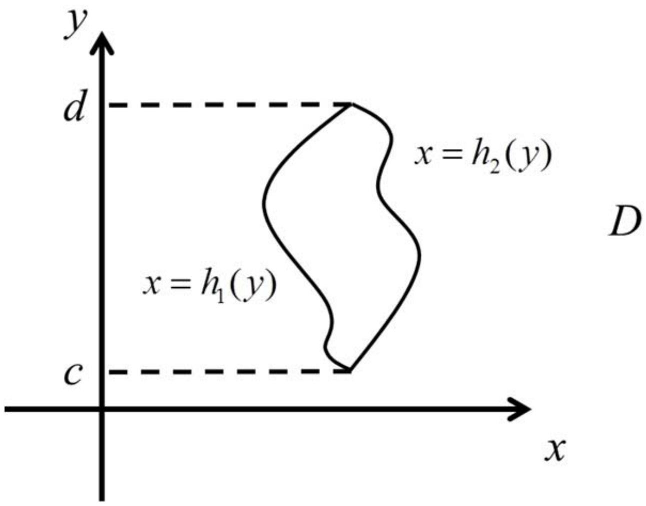
§15.3 Double Integrals over General Regions

＊General Regions

(1) Type I



(2) Type II



＊只要屋頂“不太壞”(如：沒破洞，只有些許裂縫)
則 general region 的邊界 “不太壞”(如：沒斷點)，
則一維和二維計算面積方式仍相同。

i.e.

(Type I) $\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

(Type II) $\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

Summary：

1. D 可能是

   (i) Type I

   (ii) Type II

   (iii) 是 Type I 也是 Type II

   (iv) 兩者皆不是但可表成兩種的聯集

2. If $D_1 \cup D_2 = D, D_1 \cap D_2 = \text{Boundary} \Rightarrow \iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA.$

3. Linear：$\iint_D (\alpha f + g) dA = \alpha \iint_D f dA + \iint_D g dA.$

4. If $m \le f(x, y) \le M$, then $mA(D) \le \iint_D f(x, y) dA \le MA(D).$

   Here $A(D)$ = 地基的面積.

5. $\iint_D f(x, y) dA = 0$ provided that

   (i). D 對稱 y 軸

   (ii). 固定 y，$f(x, y)$ is an odd function.

Example 1：

$$\iint_D \left(x^2 \tan x + y^3 + 4\right) dA \quad D: x^2 + y^2 = 2$$

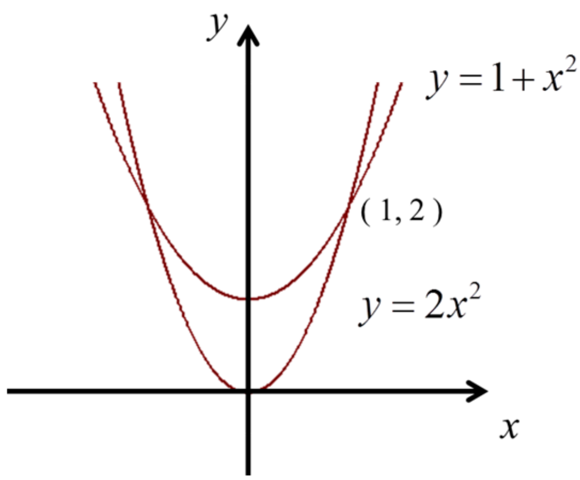
Solution：

$$\iint_D \left(x^2 \tan x + y^3 + 4\right) dA$$
$$= \iint_D x^2 \tan x dA + \iint_D y^3 dA + \iint_D 4 dA$$
$$= 8\pi$$

Example 2：

$$\iint_D (x+2y)dA, D:\begin{cases} y=2x^2 \\ y=1+x^2 \end{cases}$$

Solution：

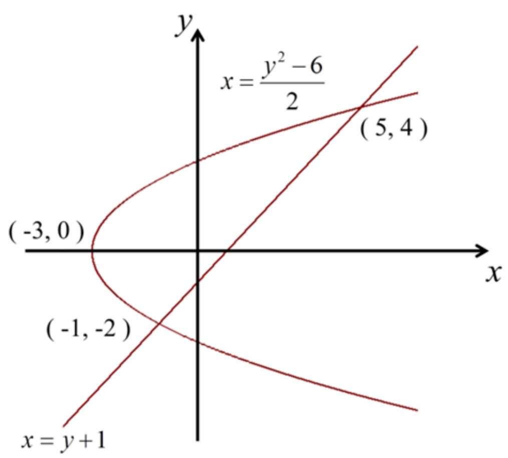


$$\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx=\frac{32}{15}$$

Example 3：

$$\iint_D xydA, D:\begin{cases} y=x-1 \\ y^2=2x+6 \end{cases}$$

Solution：

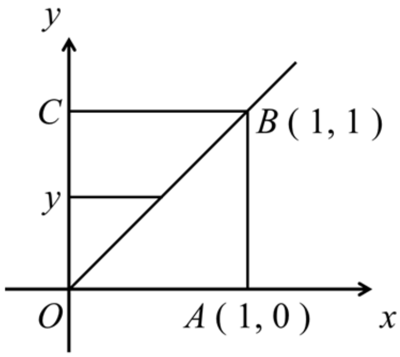


$$=\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1}xydxdy=36$$

Example 4：

$$\int_0^1\int_x^1 \sin y^2\,dydx$$

Solution：



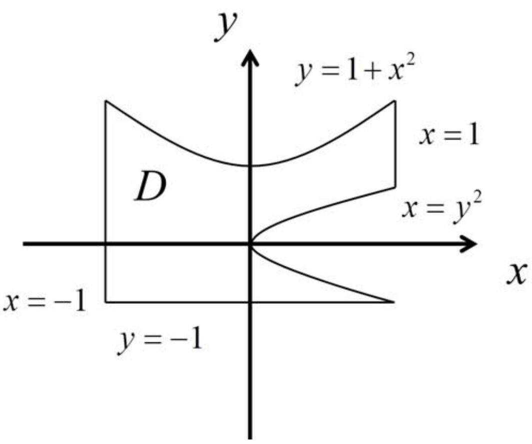
Type II：$\int_0^1\int_x^1 \sin y^2\,dydx$ (不好算)

Type I：$\int_0^1\int_0^y \sin y^2\,dxdy = \int_0^1 y\sin y^2\,dy = \int_0^1 \sin u\,du$

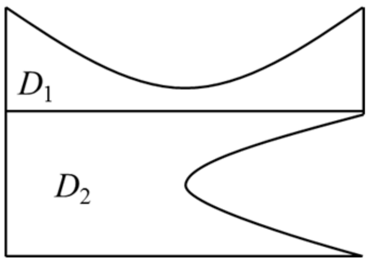
$$= -\frac{1}{2}\cos u\Big|_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}$$

Example 5：

$$\iint_D xy\,dA$$



Solution：



$$=\iint_{D_1} xy\,dA+\iint_{D_2} xy\,dA$$

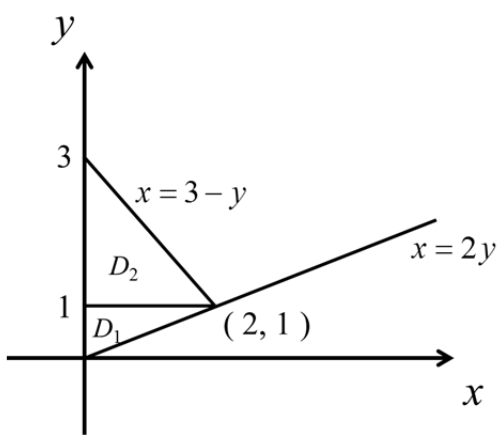
$$=\int_{-1}^{1}\int_{1}^{1+x^2} xy\,dy\,dx+\int_{-1}^{1}\int_{-1}^{y^2} xy\,dx\,dy$$

$=0$(利用對稱性)

Example 6：

$$\iint_D f(x,y)\,dA=\int_0^1\int_0^{2y} f(x,y)\,dx\,dy+\int_1^3\int_0^{3-y} f(x,y)\,dx\,dy.\text{ Find } D.$$

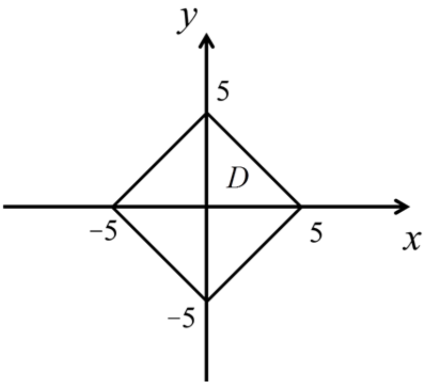
Solution：



$$D=D_1\cup D_2$$

Example 7：

$$\iint_D (2-3x+4y)\,dA$$



Solution：

(利用對稱性)

$$\iint_D (2-3x+4y)\,dA$$

$$=\iint_D 2\,dA-\iint_D 3x\,dA+\iint_D 4y\,dA$$

$$=100$$