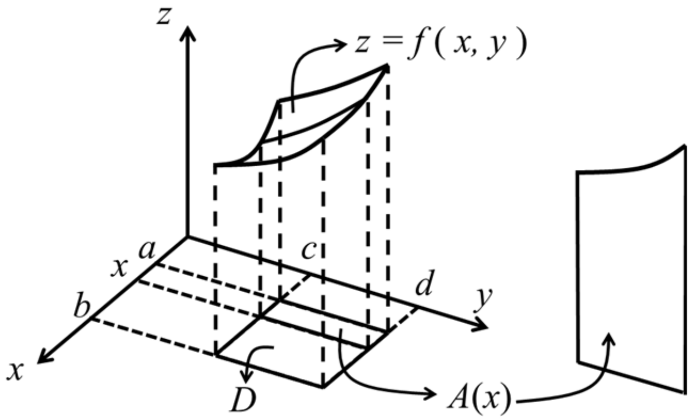
§15.2 Iterated Integrals

＊**Double Integral**：二維計算屋子體積的方法。

＊**Iterated Integral**：一維計算屋子體積的方法。(要算兩次)



(i).

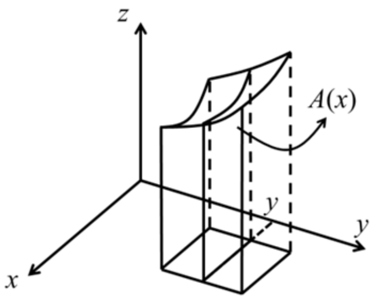
$\int_c^d f(x,y)dy = A(x)$

$=$固定x，房子的縱截面面積

$\int_a^b A(x)dx =$將所有縱截面面積連續相加

$= \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_a^b \int_c^d f(x,y)dydx$

$=$一維算法算2次.



同理可得，先固定y的縱截面面積算法而得$\int_c^d \int_a^b f(x,y)dxdy$.

Problem：

什麼樣的屋頂$\left(f(x,y)=z\right)$此三種屋子的體積算法相等？

Theorem：(Fubini's Theorem)

If f is continuous on D(屋頂沒破), then

$$\iint_D f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

Remark：

(i). In fact, f 只有些許小裂縫，三種體積算法也是一樣。

(ii). Clairaut's Theorem：

When the order of partial differentiation does not matter?

Fubini's Theorem：

When the order of iterated integrals does not matter?

Example 1：

Find $\iint_R y\sin(xy)dA,\ R=[1,2]\times[0,\pi].$

Solution：

$$\begin{aligned}\iint_R y\sin(xy)dA &= \int_0^{\pi}\int_1^2 y\sin(xy)dxdy \\ &= \int_0^{\pi}\left[-\cos xy\right]_1^2 dy \\ &= \int_0^{\pi}(-\cos 2y+\cos y)dy \\ &= -\frac{1}{2}\sin 2y\Big|_0^{\pi} + \sin y\Big|_0^{\pi} \\ &= 0(\text{有地下室，體積相消})\end{aligned}$$

Example 2：

$\iint_R (x-3y^2)dA,\ \ R=[0,2]\times[1,2].$

Solution：

$$\int_1^2\int_0^2 (x-3y^2)dxdy = \int_1^2\left(\frac{1}{2}x^2-3y^2x\right)\Big|_0^2 = \int_1^2(2-6y^2)dy = -12$$

$$\int_0^2\int_1^2 (x-3y^2)dydx = \int_0^2 (xy-y^3)\Big|_1^2 dx = -12.$$

Example 3：

$$\text{Find the volume enclose by}\begin{cases} x^2+y^2+z=16 \\ x=2 \\ y=2 \\ \text{The coordinate planes}\end{cases}.$$

Solution：

屋頂：$z = 16 - x^2 - y^2$

地基：$[0,2]\times[0,2]$

$$V = \int_0^2 \int_0^2 \left(16 - x^2 - y^2\right) dxdy = \frac{160}{3}.$$

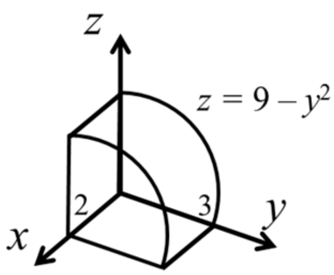
Example 4：

Find the volume of the solid in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

Solution：

屋頂：$z = 9 - y^2$

地基：$[0,2]\times[0,3]$



Example 5：

If f is continuous on $[a,b]\times[c,d]$ and $g(x,y) = \int_a^x \int_c^y f(s,t)dtds$ for $a < x < b, c < y < d$.Prove that $g_{xy} = g_{yx} = f(x,y)$.

Proof.

$$g_x \overset{FTC}{=} \int_c^y f(x,t)dt \Rightarrow g_{xy} \overset{FTC}{=} f(x,y).$$

$$g(x,y) \overset{\text{Fubini}}{=} \int_c^y \left(\int_a^x f(s,t)ds \right) dt$$

$$\Rightarrow g_y \overset{FTC}{=} \int_a^x f(s,y)ds \Rightarrow g_{yx} \overset{FTC}{=} f(x,y).$$