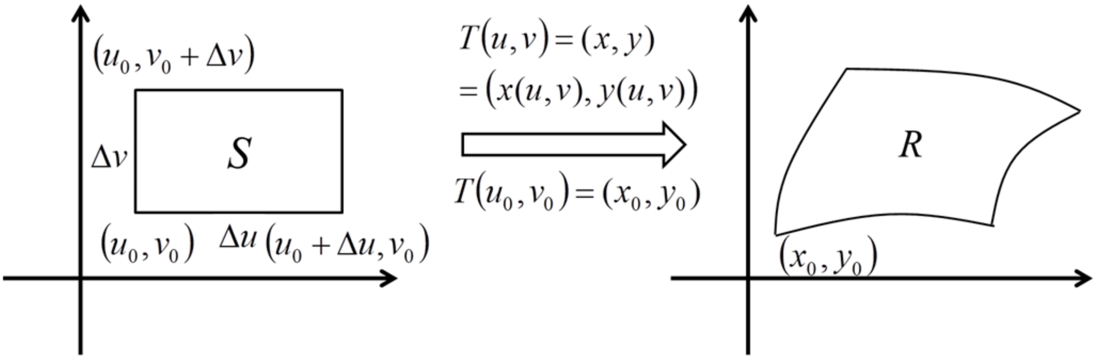
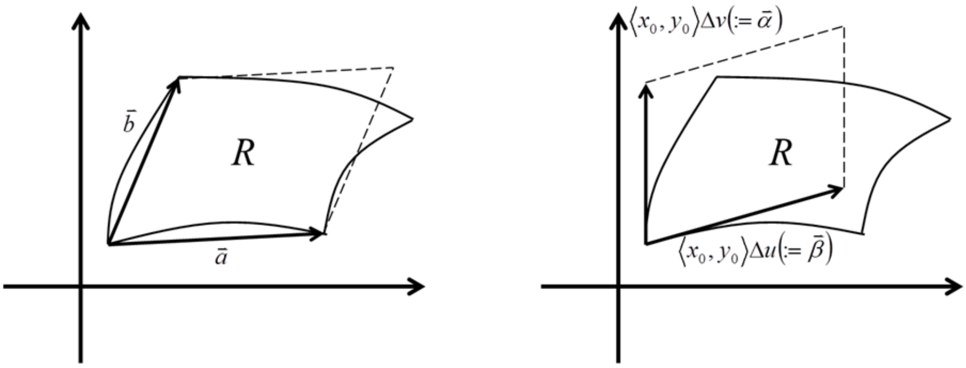
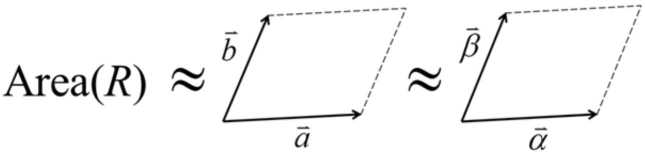
§15.10 Change of Variables in Multiple Integrals



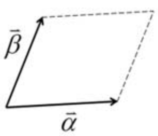




$$\langle x_u, y_u\rangle = \lim_{\Delta u\to 0}\frac{\vec{a}}{\Delta u} \Rightarrow \vec{\alpha} = \langle x_u, y_u\rangle \Delta u \approx \vec{a}$$

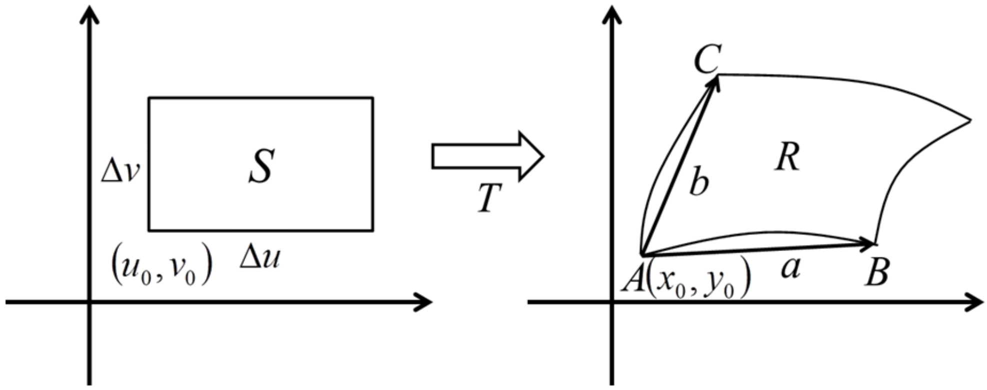
Similarly,

$$\langle x_v, y_v\rangle = \lim_{\Delta v\to 0}\frac{\vec{b}}{\Delta v} \Rightarrow \vec{b} = \langle x_v, y_v\rangle \Delta v \approx \vec{\beta}$$

Area of  $= \left|\langle x_u, y_u, 0\rangle \times \langle x_v, y_v, 0\rangle\right|$

(將$\vec{\alpha}$和$\vec{\beta}$在$x-y$平面的向量放入三維空間，以計算由$\vec{\alpha}$和$\vec{\beta}$所張出的平行四邊形面積)

$$= \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\|.$$



$T(u,v) = (x,y) = (x(u,v), y(u,v))$

$\vec{a} = \overline{AB} = (x(u_0+\Delta u, v_0), y(u_0+\Delta u, v_0)) - (x(u_0,v_0), y(u_0,v_0))$

$= (x(u_0+\Delta u, v_0) - x(u_0,v_0), y(u_0+\Delta u, v_0) - y(u_0,v_0))$

$\approx \left(\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}\right)\Delta u$

同理

$\vec{b} = \overline{AC} = \left(\dfrac{\partial x}{\partial v}\Delta v, \dfrac{\partial y}{\partial v}\Delta v\right) = \Delta v\left(\dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}\right)\Bigg|_{(u_0, v^*)}$

當$\Delta u, \Delta v$很小時

R的面積 $\approx \left|\vec{a}\times\vec{b}\right| = \left\|\begin{matrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{matrix}\right\|\Delta u\Delta v = \left\|\begin{matrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{matrix}\right\| S$的面積.

Definition：

The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix} = (\text{行列式值}).$$

＊註記：

(i). 當 $\Delta u, \Delta v$ 夠小時，轉換後的面積倍率為 Jacobian 的絕對值。

$(\Rightarrow dA = dxdy = |\text{Jacobian}|\ dudv)$

(ii). 若 T 為 Linear Transformation, 則任一區域 S(不須小)經 T 轉換後之區域 R 面積為原面積的|Jacobian|倍。

Theorem：

(i). $$\iint_R f(x,y)dA = \iint_S f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

(ii). $$\iiint_R f(x,y,z)dV = \iiint_V f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw.$$

＊註記：

(1) $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 \sin\phi.$ (2) $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r.$

Example 1：

$$\iint_R e^{\frac{x+y}{x-y}} dA$$

Solution：

Let $u = x + y$

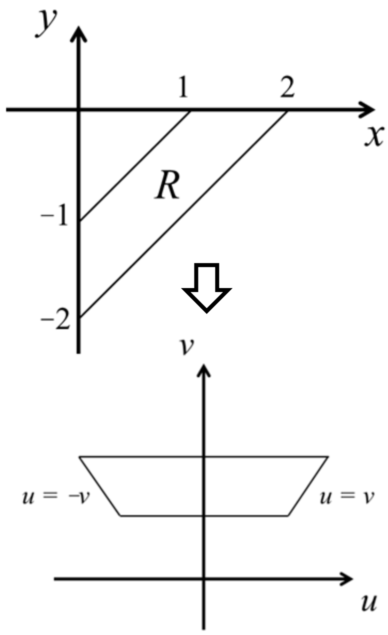
$v = x - y$

$$x = \frac{1}{2}(u+v)$$

$$y = \frac{1}{2}(u-v)$$

$$= \iint_S e^{\frac{u}{v}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \frac{1}{2}\iint_S e^{\frac{u}{v}} dA$$

$$= \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} dudv = \frac{3}{4}\left(e - e^{-1}\right)$$



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Example 2：

$$\iiint_E dV \quad E: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Set $x = au, y = bv, z = cw.$

$$= \iiint_{u^2+v^2+w^2\le 1} abc\, dV = \frac{4}{3}\pi abc.$$

Example 3：

$$\iint_R e^{x+y} dA. \quad R: |x|+|y| \le 1$$

Solution：

$$u = x + y, v = x - y$$

$$\Rightarrow x = \frac{1}{2}(u-v), y = \frac{1}{2}(u+v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\Rightarrow \iint_R e^{x+y} dA = \iint_S \frac{1}{2} e^u dA$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u du dv = e - e^{-1}.$$

Example 4：

Evaluate the integral $\iint_D e^{\frac{x+y}{x-y}} dA$, where D is the trapezoidal region with vertices (1, 0), (2, 0), (0, −2), (0, −1).

Solution：

$$\frac{3}{4}\left(e - \frac{1}{e}\right).$$