§15.1 Double Integrals over Rectangles

Problem：

$z = f(x, y) \leftrightarrow$ 房子的屋頂$(f \geq 0)$

$D = [a,b] \times [c,d] \leftrightarrow$ 房子的地基$(f \geq 0)$

$= \text{a rectangle}$

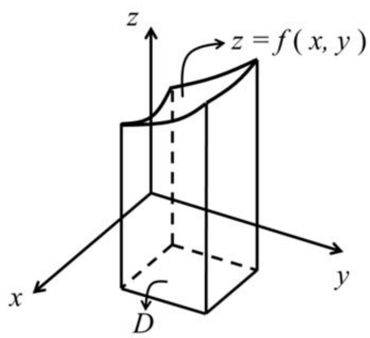
則此房子的體積(Volume)為何？

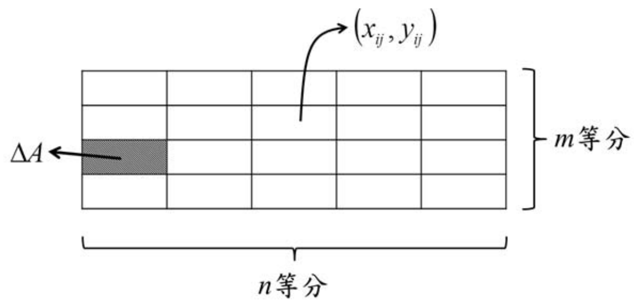
Definition：

The double integral of f over a rectangle D is

$$\lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} \underbrace{f(x_{ij}, y_{ij})}_{\text{高}} \underbrace{\Delta A}_{\text{底}} = \text{volume under the roof } z = f(x, y) \text{ on}$$

$D(f \geq 0) =$ 若$f \geq 0$,此式代表屋子的體積.





Remark：

(i). $\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}, y_{ij}\right) \Delta A$ is called double Riemann sum

= 一堆小長方形之體積和

= 這是二維的算法

(ii). Midpoint Rule：Double Riemann sum with (x_{ij}, y_{ij}) being center of the corresponding rectangle.

(iii). Average value of f on $D = \dfrac{\iint_D f(x, y)\, dA}{\text{area } D} =$ 房子的平切高度.

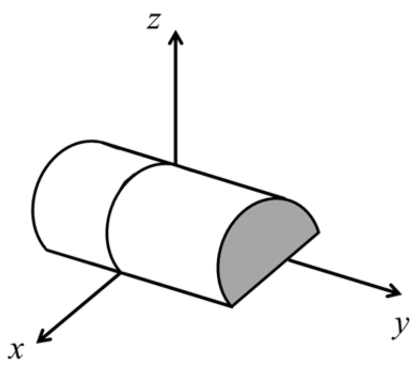
(iv). Properties of double integral.

(v). linear : $\iint_D \left(af + bg\right) dA = a \iint_D f \; dA + b \iint_D g \; dA.$

(vi). $\iint_D f \geq \iint_D g$ provided that $f \geq g$ on D.

Example 1：

屋頂：$f(x,y)=\sqrt{1-x^2}$
地基：$D=[-1,1]\times[-2,2]$.



Solution：

Evaluate $\iint_D \sqrt{1-x^2}\,dA$ = 圓柱體的一半

$$=\frac{1}{2}\pi(1)^2\times 4=2\pi.$$

Example 2：

$z=x^2+y^2$. $D=[0,1]\times[0,1]$. Using midpoint rule to compute the double Reimann sum with $n=m=2$.

Solution：

$$\iint_D (x^2+y^2)dA\approx\frac{1}{4}\left[f\left(\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{3}{4}\right)\right]$$