§14.8 Lagrange Multipliers

＊Problem：

Maximize or Minimize functions $f(x, y)$ or $f(x, y, z)$ under certain constraints.

cases	1	2	3
objective function	$f(x,y)$	$f(x,y,z)$	$f(x,y,z)$
constraints	$g(x,y)=k$	$g(x,y,z)=k$	$g(x,y,z)=k$ and $h(x,y,z)=k$

＊Lagrange Multipliers：解上述問題的一個方法

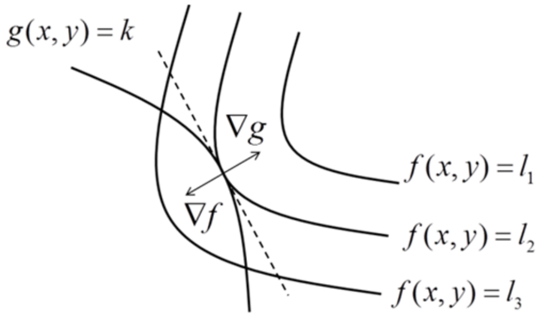
想法：

(i). 極值產生的地方：

f(變動)的 level curve or level surface 和

g(固定)的 level curve or level surface 相切.

(ii). Level curve of $f \perp \nabla f$.

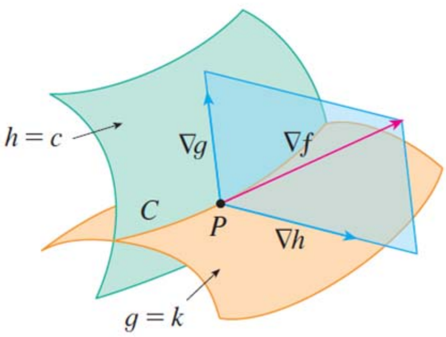


$(\text{i})+(\text{ii}) \Rightarrow \nabla f \parallel \nabla g$.

結論：Extrema occur when

(I). $\nabla f = \lambda \nabla g$ for some λ (cases 1 and 2).

(II). $\nabla f = \lambda \nabla g + \mu \nabla h$ for some λ, μ (case 3).



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Example 1：

A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such box.

Solution：

(Method I)

$$f(x,y,z)=xyz$$

constraint $g: g(x,y,z)=2xy+2yz+xz \qquad (1)$

$$\nabla f=\langle yz,xz,xy\rangle$$

$$\nabla g=\langle 2y+z,2x+2z,2y+z\rangle$$

$$\overset{\nabla f\,//\,\nabla g}{\Rightarrow}\begin{cases} yz=\lambda(2y+z) \\ xz=\lambda(2x+2z) \\ xy=\lambda(2y+x) \end{cases}$$

$$\Rightarrow\begin{cases} \dfrac{y}{x}=\dfrac{2y+z}{2x+2z} \\ \dfrac{z}{y}=\dfrac{2x+2z}{2y+x} \\ \dfrac{x}{z}=\dfrac{2y+x}{2y+z} \end{cases}\Rightarrow\begin{cases} 2y=x & (2) \\ z=2y & (3) \\ x=z \end{cases}$$

將(2),(3)代入(1)

$$\Rightarrow 4y^2+4y^2+4y^2=12$$

$$\Rightarrow y=1$$

$$\Rightarrow x=2, z=2$$

$$\Rightarrow V=4.$$

(Method II)

$$\frac{12}{3}=\frac{2xy+2yz+xz}{3}\geq\left(4x^2y^2z^2\right)^{\frac{1}{3}}$$

$$\Rightarrow 64\geq 4x^2y^2z^2$$

$$\Rightarrow 4\geq xyz$$

$$\Rightarrow \text{Maximum volume}=4.$$

Example 2：

$$f(x,y)=x^2+2y^2, g(x,y)=x^2+y^2=1.$$

Solution：

(Method I)

$$高中解法\begin{cases}1.算幾不等式\\2.柯西不等式\\3.配方法\end{cases}$$

$\Rightarrow f(x,y)=x^2+2y^2=1+y^2,(-1\le y\le 1).$

$\Rightarrow f$的最大值$=2$; f的最小值$=1.$

(Method II)

$\nabla f=\langle 2x,4y\rangle,\nabla g=\langle 2x,2y\rangle$

$\Rightarrow \begin{cases}x=\lambda x\\2y=\lambda y\end{cases}\Rightarrow x=0 \text{ or } y=0.$

$\Rightarrow (0,\pm 1)$ and $(\pm 1,0)$

$\Rightarrow \max\ f=2, \min\ f=1.$

Example 3：

$$f(x,y,z)=x+2y+3z. \text{Constraints}:\begin{cases}x-y+z=1 \ (1)\\x^2+y^2=1 \ (2)\end{cases}$$

Solution：

(Method I)

$f(x,y,z)\overset{(1)}{=}x+2y+3(1-x+y)=-2x+5y+3$

By 柯西不等式

$\Rightarrow (x^2+y^2)((-2)^2+5^2)\ge(-2x+5y)^2$

$\Rightarrow \sqrt{29}\ge -2x+5y\ge -\sqrt{29}$

$\Rightarrow \underbrace{\sqrt{29}+3}_{\max}\ge -2x+5y+3\ge \underbrace{-\sqrt{29}+3}_{\min}$

(Method II)

$$\begin{aligned}\nabla f &= \langle 1,2,3\rangle \\ \nabla g &= \langle 1,-1,1\rangle \\ \nabla h &= \langle 2x,2y,0\rangle\end{aligned} \quad \Rightarrow \begin{cases} 1 = \lambda + 2\mu x \\ 2 = -\lambda + 2\mu y \\ 3 = \lambda \end{cases}$$

$$\Rightarrow \begin{cases} 2\mu x = -2 \\ 2\mu y = 5 \end{cases} \Rightarrow \frac{x}{y} = \frac{-2}{5} \quad (\text{since } \mu \neq 0)$$

Since $x^2 + y^2 = 1 \Rightarrow x = \dfrac{\mp 2}{\sqrt{29}}, y = \dfrac{\pm 5}{\sqrt{29}}$

$$\Rightarrow z = 1 \pm \frac{7}{\sqrt{29}}$$

$$\Rightarrow \max = 3 + \sqrt{29}, \min = 3 - \sqrt{29}.$$

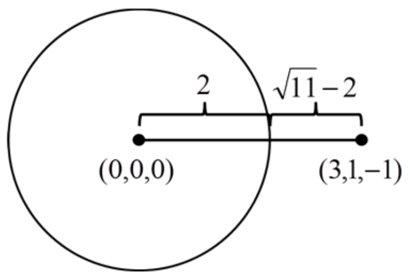
Example 4：

$x^2 + y^2 + z^2 = 4$. Find max or min of

$(x-3)^2 + (y-1)^2 + (z+1)^2$. Where those extreme occur?

Solution：

(Method I)



$$\Rightarrow \begin{cases} \min = \sqrt{11} - 2 \text{ occurs at } \left(\dfrac{6}{\sqrt{11}}, \dfrac{2}{\sqrt{11}}, \dfrac{-2}{\sqrt{11}}\right) \text{ and} \\ \max = \sqrt{11} + 2 \text{ occurs at } \left(\dfrac{-6}{\sqrt{11}}, \dfrac{-2}{\sqrt{11}}, \dfrac{2}{\sqrt{11}}\right). \end{cases}$$

Example 5：

$$\begin{cases} f(x,y,z) = x + y + z \\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1 \end{cases}$$

Solution：

$$\left(\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + \left(\sqrt{z}\right)^2 \right) \left(\left(\frac{1}{\sqrt{x}}\right)^2 + \left(\frac{1}{\sqrt{y}}\right)^2 + \left(\frac{1}{\sqrt{z}}\right)^2 \right) \geq (1+1+1)^2 = 9$$

$$\Rightarrow (x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = (x+y+z) \geq 9$$

$$\Rightarrow \min = 9.$$

Example 6：

$$\begin{cases} f(x,y,z) = x^2 + 2y^2 + 2z^2 \\ x + y + z = 1 & (1) \\ x - y + 2z = 2 & (2) \end{cases}$$

Solution：

(Method I)

$$\langle 1,1,1 \rangle \times \langle 1,-1,2 \rangle = \langle 3,-1,-2 \rangle$$

Let $z = 0 \Rightarrow \begin{cases} x + y = 1 \\ x - y = 2 \end{cases} \Rightarrow x = \dfrac{3}{2}, y = -\dfrac{1}{2}.$

$\Rightarrow (1) \cap (2)$之直線方程式為

$$\begin{cases} x = \dfrac{3}{2} + 3t \\ y = -\dfrac{1}{2} - t, \quad t \in R. & (3) \\ z = 0 - 2t \end{cases}$$

將(3)代入 f 利用配方法可得極值.

(Method II) Lagrange Multipliers

Example 7：

The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Solution：

$$\text{Max or Min}\begin{cases} f(x,y,z) = x^2 + y^2 + z^2 & \\ x + y + 2z = 2 & h \\ z = x^2 + y^2 & g \end{cases}$$

$$\nabla h = \langle 1,1,2\rangle, \nabla g = \langle 2x,2y,-1\rangle, \nabla f = \langle 2x,2y,2z\rangle$$

$$\Rightarrow \nabla f = \lambda\langle 1,1,2\rangle + \mu\langle 2x,2y,-1\rangle$$

$$\Rightarrow x = y$$

$$\Rightarrow x = \frac{1}{2} \text{ or } -1$$

$$\Rightarrow \text{Two points to check are } \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \text{ and } (-1,-1,2).$$

$$\Rightarrow \underbrace{f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \frac{3}{4}}_{\text{nearest}} \ \& \ \underbrace{f(-1,-1,2) = 6}_{\text{farthest}}.$$

Example 8：

Find the maximum of $f(x, y, z) = xy + y^2$ on the intersection of $y - x = 0$ and $x^2 + y^2 + z^2 = 4$.

Example 9：

What is the highest point on the plane curve that is given by the intersection of $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$

Example 10：

Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 9$.