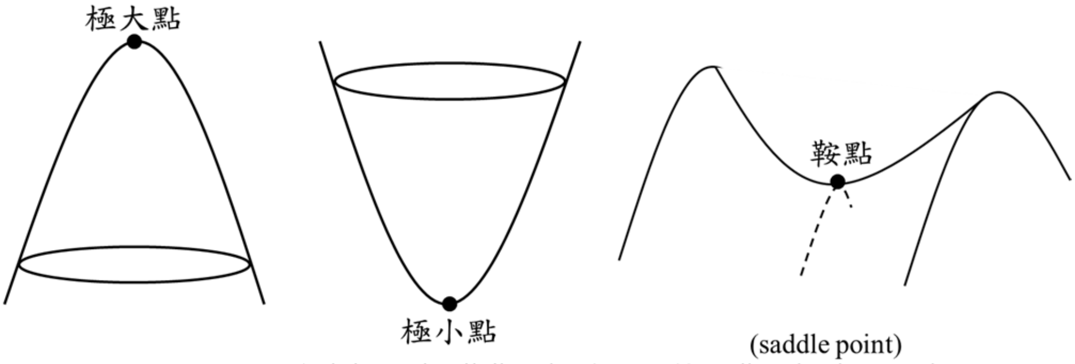
§14.7 Maximum and Minimum Values



(Maximum in all directions) (Minimum in all directions) (saddle point)
(One direction is a maximum and another direction is a minimum.)

＊單變數函數和多變數函數的差異

1. 鞍點(多變數函數)
2. 平滑單變數函數，兩個 local max $\Rightarrow$ 一個 local min.
   平滑多變數函數，兩個 local max $\nRightarrow$ 一個 local min.

＊How to find such points?

(1) Find critical points：

(i). $f_x(a,b) = f_y(a,b) = 0$

(ii). f_x or f_y does not exist at (a, b).

(2) 2^{nd} Derivative Tests：

Let $\alpha = f_{xx}(a,b), \beta = f_{xy}(a,b) = f_{yx}(a,b), \gamma = f_{yy}(a,b)$

Set $D = \begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} = \alpha\gamma - \beta^2.$

(i). $D > 0, \alpha > 0$ (or $\gamma > 0$)
$\Rightarrow f(a,b)$ local minimum and (a,b) is a minimum point.

(ii). $D > 0, \alpha < 0$(or $\gamma < 0$) $\Rightarrow f(a,b)$ local maximum.

(iii). $D < 0 \Rightarrow (a,b)$ is a saddle point.

(iv). For any other cases, the test fails.

Why?

(2)-(i)的條件中

$\alpha > 0 \Rightarrow$ 在x軸方向$(a,b,f(a,b))$是凹向上.

$D > 0 \Rightarrow$ 其他所有方向也凹向上.

Proof.

$u = \langle h,k \rangle$.

$D_u f(a,b) = f_x h + f_y k$.

$D_u^2 f = D_u(D_u f) = \left(f_{xx}h + f_{xy}k\right)h + \left(f_{yx}h + f_{yy}k\right)k$

$= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$

$= ah^2 + 2\beta hk + \gamma k^2$

$= \alpha\left(h + \dfrac{\beta k}{\alpha}\right)^2 + \dfrac{k^2}{\alpha}(\alpha\gamma) - \beta^2.$

If $\alpha > 0, D > 0 \Rightarrow D_u^2 f > 0$ for any direction u.

$\Rightarrow$ All directions at $(a,b,f(a,b))$ are concave up.

Example 1：

$f(x,y) = x^4 + y^4 - 4xy + 1$. Find local max, min and saddle point.

Solution：

$$\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{cases}$$

$\Rightarrow (0,0), (1,1), (-1,-1)$ are critical points.

$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$

$D(x,y) = 144x^2y^2 - 16$

$D(0,0) = -16 < 0.$

$\Rightarrow (0,0)$ is a saddle point.

$D(1,1) = 128 > 0,\ f_{xx}(1,1) = 12 > 0.$

$\Rightarrow f(1,1) = -1$ is a local min.

$D(-1,-1) = 128 > 0,\ f_{xx}(-1,-1) = 12 > 0.$

$\Rightarrow f(-1,-1) = -1$ is also a local min.

Theorem：

f is continuous on a closed and bounded set D.

⇒(i) There exist absolute minimum and absolute maximum.

(ii) Those extreme points occur at the critical points of f or the boundary of D.

Example 2：

Find the saddle points of the function $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$

Example 3：

Find the local maximum and minimum and saddle points of the function

$f(x, y) = x^3 + y^2 - 3xy + 5$

Example 4：

$f(x, y) = 2x^2 + x + y^2 - 2, x^2 + y^2 \leq 4.$ Find absolute extreme.

Solution：

$$\begin{matrix} f_x = 4x + 1 \\ f_y = 2y \end{matrix} \Rightarrow x = -\frac{1}{4}, y = 0.$$

(x, y)	$\left(-\frac{1}{4}, 0\right)$	$x^2 + y^2 = 4$
$f(x, y)$	$-\frac{17}{8}$	$x^2 + x + 2$ $-2 \leq x \leq 2$
values	$-\frac{17}{8}$	$4, 8, \frac{7}{4}$

$\max = 8$ occurs at $x = 2, y = 0.$

$\min = -\frac{17}{8}$ occurs at $x = -\frac{1}{4}, y = 0.$

Example：

$f(x,y) = x^2 - 2xy + 2y$ on $0 \le x \le 3, 0 \le y \le 2$.

Find absolute max and min.

Solution：

$$\begin{aligned} f_x &= 2x - 2y \\ f_y &= -2x + 2 \end{aligned} \Rightarrow x = 1, y = 1.$$

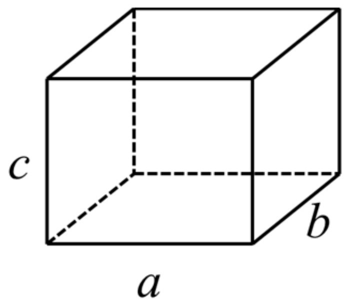
(x,y)	$(1,1)$	$x=0$ $0 \le y \le 2$	$x=3$ $0 \le y \le 2$	$y=0$ $0 \le x \le 3$	$y=2$ $0 \le x \le 3$
$f(x,y)$	1	$2y$	$9-4y$	x^2	$(x-2)^2$
value	1	0,4	1,9	0,9	0,4

0：absolute minimum.

9：absolute maximum.

Example：

A box with open top with constraint：



$ab + 2bc + 2ac = 12$

Find max volume.

Solution：

$$\frac{ab + 2bc + 2ac}{3} \ge \sqrt[3]{4a^2b^2c^2}$$

$$64 \ge 4a^2b^2c^2$$

$$4 \ge abc$$

Maximum volume = 4 when $ab = 2bc = 2ac$

$\Rightarrow a = 2, b = 2, c = 1.$

Example：

(x,y,z) satisfies $x + 2y + 3z = 6$. Find max $V = xyz$.

Solution：

$$\frac{6}{3} \ge \sqrt[3]{6xyz} \Rightarrow xyz \le \frac{4}{3}$$

Max $V = \frac{4}{3}$ when $x = 2y = 3z \Rightarrow x = 2, y = 1, z = \frac{2}{3}$.