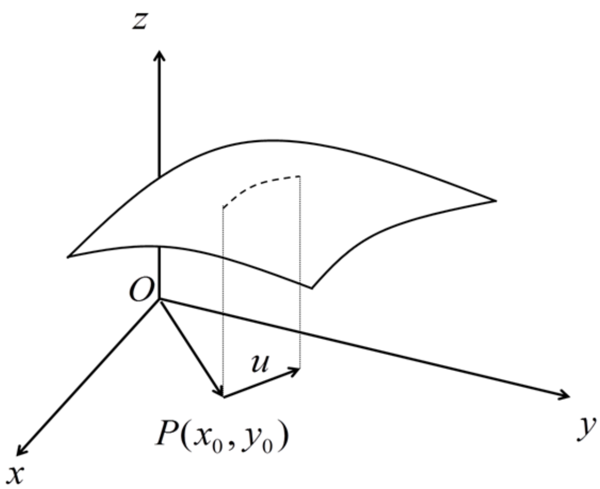
§14.6 Directional Derivatives and Gradient Vector

＊方向導數：過點(x_0, y_0)，沿著$u = \langle a,b \rangle, a^2 + b^2 = 1$,方向的微分



Definition：過點(x_0, y_0)，沿著 u 方向的微分

$$u = \langle a,b \rangle, a^2 + b^2 = 1$$

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h \to 0} \frac{f\left(\overrightarrow{OP} + hu\right) - f(\overrightarrow{OP})}{h} \\ &= \lim_{h \to 0} \frac{f\left(x_0 + ha, y_0 + hb\right) - f(x_0, y_0)}{h} \\ &= g'(0) \qquad \text{, where } g(h) = f(x_0 + ha, y_0 + hb) \end{aligned}$$

Remark：

1. $f_x = Df_i, \; i = \langle 1,0 \rangle$

2. $f_y = Df_j, \; j = \langle 0,1 \rangle$

Theorem：

$$\begin{aligned} D_u f(x_0, y_0) &= \left\langle f_x(x_0, y_0), f_y(x_0, y_0) \right\rangle \bullet \langle a,b \rangle \\ &= af_x(x_0, y_0) + bf_y(x_0, y_0) \end{aligned}$$

Proof.

Let $g(h) = f(x_0 + ah, y_0 + bh)$
Then $D_u f(x_0, y_0) = g'(0)$

$g'(h) = f_x a + f_y b$
$\Rightarrow g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$

Remark：

If the direction derivative of f at (x_0, y_0) in any direction u, then f is not necessary differentiable at (x_0, y_0).

＊The gradient vector (梯度向量) of f at (x_0, y_0)

Definition：

$$\left\langle f_x(x_0, y_0), f_y(x_0, y_0) \right\rangle =: \nabla f$$

$$D_u f(x_0 y_0) = \nabla f \bullet u$$

Example 1：

$f(x, y) = x^3 - 3xy + 6y^2$. Find the directional derivative of f at (1,2) in the dirction of the vector $\left\langle \sqrt{3}, 1 \right\rangle$.

Solution：

$$D_u f(1,2) = \left\langle 3x^2 - 3y, -3x + 12y \right\rangle \Big|_{\substack{x=1 \\ y=2}} \bullet \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

$$= \frac{-3\sqrt{3} + 21}{2}.$$

Example 2：

A(1, 2), B(4, 6), C(5, −1), D(6, 14). Let f be a function of two variable that has continuous partial derivatives.

Let $u_1 = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}, u_2 = \frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|}, u_3 = \frac{\overrightarrow{AD}}{\left|\overrightarrow{AD}\right|}$.

If $D_{u_1}(1,2) = 1, D_{u_2}(1,2) = -1$, find $D_{u_3}(1,2) = ?$

＊Facts

1. $D_u f = \nabla f \bullet u = f$在$u$方向的變化率

2. $\max_u D_u f = |\nabla f|$ when $u // \nabla f$ (same direction)

   $\because \nabla f \bullet u = |\nabla f||u|\cos\theta,\ |u| = 1$
   $= |\nabla f|$ if $\theta = 0$.

3. $\min_u D_u f = -|\nabla f|$ when $u // \nabla f$ (opposite direction)

4.

(i). $\nabla f \perp$ level curve.

(ii). $\nabla f \perp$ any curves on the level surface.

Proof.

(i). Consider $\Gamma: f\big(x(t), y(t)\big) = C$.

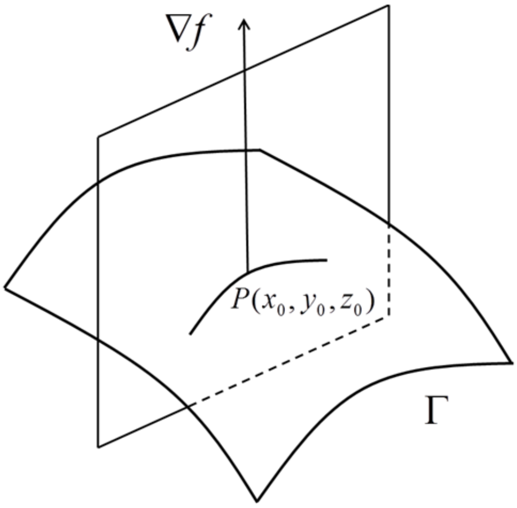
$$\Rightarrow f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

$$\Rightarrow \nabla f \bullet \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0$$

(ii). Consider $\Gamma: f(x, y, z) = C$

Let $f\big(x(t), y(t), z(t)\big) = C$ be any curve on the level surface $\Gamma: f(x, y, z) = C$.

Then $\nabla f \bullet \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = 0$



5. Consider surface $\Gamma: z = f(x, y)$. Let $g(x, y, z) = f(x, y) - z = 0$

則 Γ 可當成 g 函數的一個 Level surface. 此 g 函數的 gradient 為

$\nabla g = \langle f_x, f_y, -1\rangle$.(即將二維函數看成三維函數的等高曲面).

由(4. ii.) ∇g 為過點 $P(x_0, y_0, z_0)$ 和 Γ 相切的平面的法向量.

Example 3：

Given a level surface：$\frac{x^2}{4}+\frac{y^2}{1}+\frac{z^2}{9}=3.$ 求過(−2, 1, −3)且與此等高面相切的切面方程式.

Solution：

$$n=\left\langle \frac{x}{2},2y,\frac{2z}{9}\right\rangle\Big|_{\substack{x=-2\\y=1\\z=-3}}=\left\langle -1,2,-\frac{2}{3}\right\rangle$$

切面方程式$\left\langle -1,2,-\frac{2}{3}\right\rangle \bullet \langle x+2,\ y-1,\ z+3\rangle=0$

$$\Rightarrow -(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$$

Example 4：

$\begin{cases} f: z=x^2+y^2 \\ g: 4x^2+y^2+z^2=9 \end{cases}$ 求過(−1, 1, 2)和其交集的曲線相切的切線方程式.

Solution：

此切線的方向和$\nabla f,\nabla g$垂直

$\Rightarrow$ 切線的方向為$\nabla f\times\nabla g$.

$$\nabla f=\langle 2x,2y,-1\rangle=\langle -2,2,-1\rangle$$

$$\nabla g=\langle 8x,2y,2z\rangle=2\langle -4,1,2\rangle$$

$$\langle -2,2,-1\rangle\times\langle -4,1,2\rangle=\langle 5,8,6\rangle$$

$$\Rightarrow \frac{x+1}{5}=\frac{y-1}{8}=\frac{z-2}{6}.$$